Loop gravity from a spinor action

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Outline of the talk

I present an action for discretized gravity with spinors as the fundamental configuration variables. The theory has a Hamiltonian and local gauge symmetries. Generic solutions represent twisted geometries, and have curvature – there is a deficit angle around triangles.

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References:

*WMW, New action for simplicial gravity in four dimensions, (2014), arXiv:1407.0025.
*WMW, One-dimensional action for simplicial gravity in three dimensions, accepted for publication in Phys. Rev. D (2014), arXiv:1402.6708.
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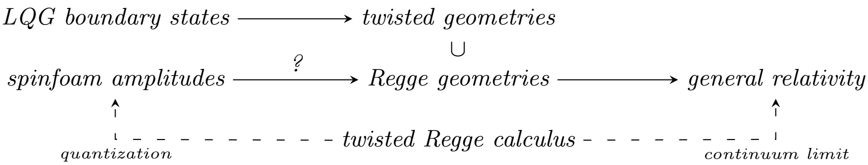
Motivation



Tension between LQG kinematics and dynamics

- Kinematics: The LQG boundary states represent twisted geometries: Every tetrahedron has a unique volume, and every triangle has a unique area, yet there are no unique edge lengths.
- Dynamics: Spinfoam gravity provides us with the transition amplitudes between generic boundary states.
- A conceptual tension: We always try to find just Regge gravity in the semi-classical limit. Yet, our kinematical framework is more general: Twisted geometries are less restrictive than Regge discretizations.
- Key question: Can we formulate the dynamics of discretized gravity in terms of twisted geometries?

New action for simplicial gravity in first-order spin-variables

Plebański principle

The BF action is topological, and determines the symplectic structure of the theory:

$$S_{\mathrm{BF}}[\Sigma, A] = \frac{\hbar}{2{\ell_{\mathrm{P}}}^2} \int_M \left(*\, \Sigma_{\alpha\beta} - \beta^{-1}\Sigma_{\alpha\beta}\right) \wedge F^{\alpha\beta}[A] \equiv \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta}. \qquad (1)$$

General relativity follows from the simplicity constraints added to the action:

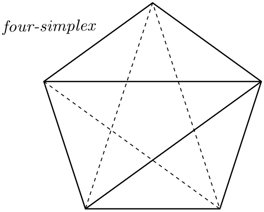
$$\Sigma^{\alpha\beta} \wedge \Sigma^{\mu\nu} \propto \epsilon^{\alpha\beta\mu\nu}. \qquad (2)$$

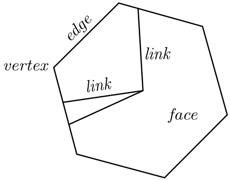
With the solutions:

$$\Sigma^{\alpha\beta} = \begin{cases} \pm e_\alpha \wedge e_\beta, \\ \pm *\, (e_\alpha \wedge e_\beta). \end{cases} \qquad (3)$$

Notation:

- $\alpha, \beta, \gamma \ldots$ are internal Lorentz indices.
- $\Sigma^\alpha{}_\beta$ is an $\mathfrak{so}(1,3)$-valued two-form.
- $A^\alpha{}_\beta$ is an $SO(1,3)$ connection, with $F^\alpha{}_\beta = \mathrm{d}A^\alpha{}_\beta + A^\alpha{}_\mu \wedge A^\mu{}_\beta$ denoting its curvature.
- e^α is the tetrad, diagonalizing the four-dimensional metric $g = e_\alpha \otimes e^\alpha$.
- ${\ell_{\mathrm{P}}}^2 = 8\pi\hbar/Gc^3$, and β is the Barbero–Immirzi parameter.





We can write the discretized BF action as a sum over the two-dimensional simplicial faces $f_1, f_2, \ldots$:

$$\boxed{\begin{aligned} S_{\mathrm{BF}}[Z_{f_1}, Z_{f_2}, \ldots; \underset{\sim}{Z}_{f_1}, \underset{\sim}{Z}_{f_2}, \ldots; \zeta_{f_1}, \zeta_{f_2}, \ldots; \Lambda_{e_1}, \Lambda_{e_2}, \ldots] &= \sum_{f:\mathrm{faces}} S_f \\ = \sum_{f:\mathrm{faces}} \oint_{\partial f} \Big[\pi^f_A D\omega^A_f - \underset{\sim}{\pi}^f_A \mathrm{d}\underset{\sim}{\omega}^A_f + \zeta_f \big(\pi^f_A \omega^A_f - \underset{\sim}{\pi}^f_A \underset{\sim}{\omega}^A_f\big)\Big] &+ \mathrm{cc.} \end{aligned}} \tag{4}$$

Notation:

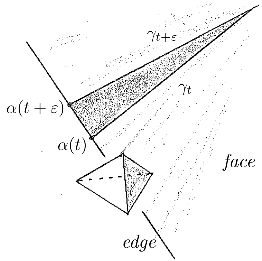
- $A, B, C, \ldots$ are spinor indices, and $\mathrm{cc.}$ denotes complex conjugation.
- Each face f carries two twistors: $Z_f, \underset{\sim}{Z}_f : \partial f \to \mathbb{T} \simeq \mathbb{C}^4$, $Z = (\bar{\pi}_{A'}, \omega^A)$.
- $\zeta_f : \partial f \to \mathbb{C}$ is a Lagrange multiplier imposing the constraint $\varDelta_f = \underset{\sim}{\pi}_A \underset{\sim}{\omega}^A - \pi_A \omega^A$.
- D is the covariant differential, $\dot{e}$ an edge's tangent vector: $\dot{e} \lrcorner D\pi^A = \dot{\pi}^A + [\Lambda_e]^A{}_B \pi^B$.

- Step 1: Discretize the action:

$$S_{\mathrm{BF}}[\Sigma, A] = \int_M \Pi_{\alpha\beta} \wedge F^{\alpha\beta} \approx \sum_{f:\mathrm{faces}} \int_{\tau_f} \Pi_{\alpha\beta} \int_f F^{\alpha\beta} \equiv \sum_{f:\mathrm{faces}} S_f .$$

- Step 2: Define the smeared flux:

$$\Pi_f^{\alpha\beta}(t) = \int_{\tau_f} \mathrm{d}x\,\mathrm{d}y\,[h_{\gamma(t,x,y)}]^{\alpha}{}_{\mu}[h_{\gamma(t,x,y)}]^{\beta}{}_{\nu}\left[\Pi_{p(x,y)}(\partial_x, \partial_y)\right]^{\mu\nu}.$$



- Step 3: Employ the non-Abelian Stoke’s theorem:

$$\int_{\gamma_t} \mathrm{d}z\, h^{-1}_{\gamma_t(z)} F_{\gamma_t(z)}(\partial_z, \partial_t) h_{\gamma_t(z)} = h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)},$$

to eventually find the one-dimensional action:

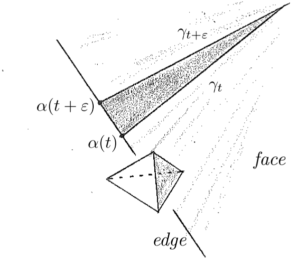
$$S_f = -\int_{\partial f} \mathrm{d}t \left[h^{-1}_{\gamma_t(1)} \frac{D}{\mathrm{d}t} h_{\gamma_t(1)}\right]_{\alpha\beta} \Pi_f^{\alpha\beta}(t).$$

Key ideas of the proof, 2/2

- Step 4: Introduce spinors to diagonalize both holonomies and fluxes:

$$\Pi_f^{\alpha\beta}(t) = \frac{1}{2}\bar{\epsilon}^{A'B'}\omega_f^{(A}(t)\pi_f^{B)}(t) + \mathrm{cc.},$$

$$[h_{\gamma_t}]^A{}_B = \mathrm{Pexp}\Big(-\int_{\gamma_t} A\Big)^A{}_B = \frac{\underset{\sim}{\omega}{}_f^A(t)\pi^f_B(t) - \underset{\sim}{\pi}{}^A_f(t)\omega^f_B(t)}{\sqrt{E_f(t)}\sqrt{\underset{\sim}{E}{}_f(t)}}.$$



We also need the area-matching constraint:

$$\varDelta_f := \underset{\sim}{\pi}{}^f_A\underset{\sim}{\omega}{}^A_f - \pi^f_A\omega^A_f \equiv \underset{\sim}{E}{}_f(t) - E_f(t).$$

Putting the pieces together yields the face action:

$$S_f[Z, \underset{\sim}{Z}, A, \zeta] =$$
$$= \int_{\partial f} \mathrm{d}t\Big[\pi_A\frac{D}{\mathrm{d}t}\omega^A - \underset{\sim}{\pi}{}_A\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{\omega}{}^A - \zeta\varDelta\Big] + \mathrm{cc.} \qquad (6)$$

Linear simplicity constraints

Instead of discretizing the quadratic simplicity constraints

$$\Sigma_{\alpha\beta} \wedge \Sigma_{\mu\nu} \propto \epsilon_{\alpha\beta\mu\nu}, \tag{7}$$

we will use the linear simplicity constraints:

For every tetrahedron T_e (dual to an edge e) there exist an internal future-oriented four-normal n_e^α such that the fluxes through its four bounding triangles τ_f (dual to a face f: $e \subset \partial f$) annihilate n_e^α:

$$\int_{\tau_f} \Sigma_{\alpha\beta} n_e^\beta = 0. \tag{8}$$

The spinorial parametrization turns the simplicity constraints into the following complex conditions:

$$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \mathrm{cc.} \stackrel{!}{=} 0, \tag{9a}$$

$$W_{ef} = n_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \stackrel{!}{=} 0. \tag{9b}$$

Adding the simplicity constraints

- The simplicity constraints reduce the $SO(1,3)$ spin connection $A^{\alpha}{}_{\beta}$ to the $SU(2)_n$ Asthekar–Barbero connection:

$$\mathcal{A}^{\alpha} = n^{\mu}\Big[\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\rho}A^{\nu}{}_{\rho} + \beta A^{\alpha}{}_{\mu}\Big]. \qquad (10)$$

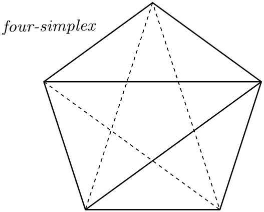
- We introduce Lagrange multipliers $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$ and get the following constrained action for each face in the discretization:

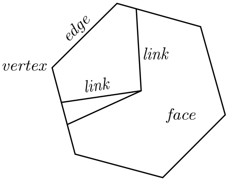
$$S_{\text{face}}[Z,\underset{\sim}{Z}|\zeta,z,\lambda|\mathcal{A},n] = \oint_{\partial f}\Big(\pi_A \mathcal{D}\omega^A - \underset{\sim}{\pi}_A \mathrm{d}\underset{\sim}{\omega}^A - \zeta\big(\underset{\sim}{\pi}_A\underset{\sim}{\omega}^A - \pi_A\omega^A\big) + \\ - \frac{\lambda}{2}\Big(\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi_A\omega^A + \mathrm{cc.}\Big) - z\, n^{AA'}\pi_A\bar{\omega}_{A'}\Big) + \mathrm{cc.}, \qquad (11)$$

  where $\mathcal{D}\pi^A = \mathrm{d}\pi^A + \mathcal{A}^{\alpha}\tau^{A}{}_{B\alpha}\pi^B$ is the $SU(2)_n$ covariant differential.

- Problem: There is no term in the action that would determine the t-dependence of the normal n^{α}_e along the edges $e(t)$.
- We now have to make a proposal.

Four-dimensional closure constraint





Any proposal for the dynamics of the time normals must respect the closure constraint at the vertices (four-simplices):

We define the volume-weighted four-normal:

$$p^e_\alpha = n^e_\alpha \mathrm{Vol}(e). \tag{12}$$

At every four simplex we have the closure constraint:

$$\sum_{\substack{\text{outgoing edges } e \\ \text{at } v}} p^e_\alpha = \sum_{\substack{\text{incoming edges } e \\ \text{at } v}} p^e_\alpha . \tag{13}$$

Notation:

- $\mathrm{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \mathrm{cc}.$

Any proposal for the dynamics of the time-normals
- must respect the four-dimensional closure constraint, and
- be consistent with all symmetries of the action.

The following action fulfills these requirements:

$$S_{\mathrm{edge}}[X, p|N, \mathrm{Vol}(e)] = \int_e \Big(p_\alpha \mathrm{d}X^\alpha - \frac{N}{2}\big(p_\alpha p^\alpha + \mathrm{Vol}^2(e)\big)\Big). \tag{14}$$

We just need an additional boundary term at the vertices:

$$S_{\mathrm{vertex}}[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}] = \sum_{e:e\ni v} \big(Y_v^\alpha - X_{ev}^\alpha\big) v_\alpha^{ev}. \tag{15}$$

Where N is a Lagrange multiplier imposing the mass-shell condition:

$$C := \frac{1}{2}\Big(p_\alpha p^\alpha + \mathrm{Vol}^2(e)\Big) \stackrel{!}{=} 0. \tag{16}$$

Putting the pieces together – defining the action

Adding the face, edge and vertex contributions gives us a proposal for an action for discretized gravity in first-order variables:

$$S_{\text{spin-Regge}} = \sum_{f:\text{faces}} S_{\text{face}}\big[Z_f, \underset{\sim}{Z}_f \big| \zeta_f, z_f, \lambda_f \big| \mathcal{A}_{\partial f}, n_{\partial f}\big] + \\ + \sum_{e:\text{edges}} S_{\text{edges}}\big[X_e, p_e \big| N_e, \mathrm{Vol}(e)\big] + \\ + \sum_{v:\text{vertices}} S_{\text{vertex}}\big[Y_v, \{X_{ev}\}_{e\ni v}, \{v_{ev}\}_{e\ni v}\big]. \tag{17}$$

Notation:

- Z_f and $\underset{\sim}{Z}_f$ are the twistors $Z_f : \partial f \rightarrow \mathbb{T} \simeq \mathbb{C}^4$ parametrizing the $SL(2,\mathbb{C})$ holonomy-flux variables.
- ζ_f, λ_f and z_f are Lagrange multipliers imposing the area-matching constraint and simplicity constraints respectively.
- $\mathcal{A}$ is the $SU(2)_n$ Ashtekar–Barbero connection along the edges of the discretization.
- n denotes the time normal of the elementary tetrahedra.
- p_e is the volume-weighted time-normal, of the tetrahedron dual to the edge e.
- $\mathrm{Vol}(e)$ denotes the corresponding three-volume.
- N is a Lagrange multiplier imposing the mass-shell condition $C = 0$.

Hamiltonian formulation, twisted geometries and curvature

Three immediate tests for the model

1. Is there a Hamiltonian formulation of the dynamics of the theory?
2. What kind of four-dimensional geometries do the equations of motion generate?
3. Does the model have curvature?

Hamiltonian formulation

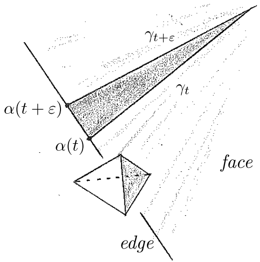
The Hamiltonian:

$$H = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \Big(\zeta^f \varDelta_f + \bar\zeta^f \bar\varDelta_f + z^f W_{ef} + \bar z^f \bar W_{ef} + \lambda^f V_f\Big) + N C_e, \quad (18)$$

generates the t-evolution along the edges of the discretization:

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega^A_f = \{H, \omega^A_f\}. \quad (19)$$

The fundamental Poisson brackets are:



$$\{p^e_\alpha, X^\beta_e\} = \delta^\beta_\alpha,$$

$$\{\pi^f_A, \omega^B_{f'}\} = +\delta_{ff'}\delta^B_A, \qquad \{\bar\pi^f_{A'}, \bar\omega^{B'}_{f'}\} = +\delta_{ff'}\delta^{B'}_{A'},$$

$$\{\underset{\sim}{\pi}{}^f_A, \underset{\sim}{\omega}{}^B_{f'}\} = -\delta_{ff'}\delta^B_A, \qquad \{\underset{\sim}{\bar\pi}{}^f_{A'}, \underset{\sim}{\bar\omega}{}^{B'}_{f'}\} = -\delta_{ff'}\delta^{B'}_{A'}.$$

Dirac analysis

- The Hamiltonian preserves all constraints provided $z_f = 0$.
- There are no secondary constraints.

Physical Hamiltonian

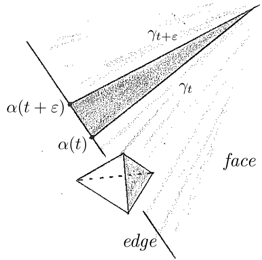
$$H_{\text{phys}} = \mathcal{A}^\alpha G_\alpha + \sum_{f:\partial f \supset e} \Big(\zeta^f \varDelta_f + \bar{\zeta}^f \bar{\varDelta}_f + \lambda^f V_f \Big) + NC. \qquad (21)$$

second-class simplicity constraint:	$W_{ef} = n_e^{AA'} \pi^f_A \bar{\omega}^f_{A'} \stackrel{!}{=} 0,$
first-class simplicity constraint:	$V_f = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi^f_A \omega^A_f + \text{cc.} \stackrel{!}{=} 0,$
area-matching condition (first-class):	$\varDelta_f = \underset{\sim}{\pi}{}^f_A \underset{\sim}{\omega}{}^A_f - \pi^f_A \omega^A_f \stackrel{!}{=} 0,$
mass-shell condition (first-class):	$C_e = \frac{1}{2}\big(p^e_\alpha p^\alpha_e + \text{Vol}^2(e)\big) \stackrel{!}{=} 0,$
$SU(2)_n$ Gauß constraint (first-class):	$G^e_\alpha = \sum_{f:\partial f \supset e} \tau^{AB}{}_\alpha \omega^f_A \pi^f_B + \text{cc.}$

Notation:

- ${\tau^A}_{B\alpha}$ are the $SU(2)_n$ generators: $[\tau_\alpha, \tau_\beta] = n^\mu \epsilon_{\mu\alpha\beta}{}^\nu \tau_\nu$.
- $\text{Vol}(e) \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$, with e.g.: $L^1_\alpha = -\tau^{AB}{}_\alpha \omega^{f_1}_A \pi^{f_1}_B + \text{cc.}$

What kind of four-dimensional geometries does the Hamiltonian generate?



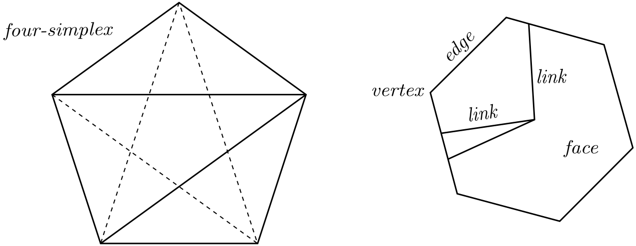
- The simplicity constraints guarantee that the fluxes $\int_{\tau_f} \Sigma_{\alpha\beta}$ define planes in internal Minkowski space.
- The Gauß constraint tells us that these planes close to form a tetrahedron.
- The physical Hamiltonian H_{phys} deforms the shape of the tetrahedron.

The Hamiltonian generates twisted geometries, the relevant term is the mass-shell condition:

$$C = \frac{1}{2}\big(p_\alpha p^\alpha + \mathrm{Vol}^2\big). \tag{23}$$

$\mathrm{Vol}^2 \propto \frac{2}{9} n_\alpha \epsilon^{\alpha\beta\mu\nu} L^1_\beta L^2_\mu L^3_\nu$ preserves the area of the four bounding triangles, and the volume of the tetrahedron, yet it does not preserve the tetrahedron's shape – the Hamiltonian generates a shear.

*E Bianchi, HM Haggard, Bohr-Sommerfeld Quantization of Space, Phys.Rev. D 86 (2012), arXiv:1208.2228.



Inter-tetrahedral angles:

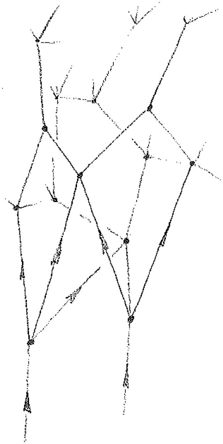
$$\cosh \Xi_{vf} = -\eta^{\mu\nu} n^{e}_{\mu} n^{e'}_{\nu}, \quad \text{with: } e \cap e' = v, \text{ and: } e,\, e' \subset \partial f. \tag{24}$$

Deficit angle around a triangle:

$$\Xi_f := \sum_{v\colon \text{vertices in } f} \Xi_{vf} = \frac{2}{\beta^2 + 1} \oint_{\partial f} \lambda_f. \tag{25}$$

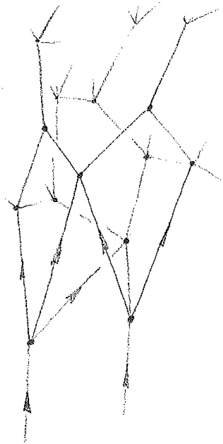
Conclusion

Basic ideas



- I have proposed an action for discretized gravity in first-order spin variables.
- The action is an integral over the entire system of edges, an action for a one-dimensional branched manifold.

The relevance of the model



- The system has a finite-dimensional phase space. The Hamiltonian is a sum over constraints, and preserves both first- and second-class constraints.
- The Hamiltonian generates twisted geometries, that appear in the semi-classical limit of loop quantum gravity.
- Going once around a triangle we pick up a deficit angle, hence the model has curvature.

Thank you for the attention, and thank you for the invitation.

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