On the K-theoretic classification of topological phases of matter

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Kitaev's Periodic Table of topological insulators and superconductors

TABLE 1. Classification of free-fermion phases with all possible combinations of the particle number conservation (Q) and time-reversal symmetry (T). The $\pi_0(C_q)$ and $\pi_0(R_q)$ columns indicate the range of topological invariant. Examples of *topologically nontrivial* phases are shown in parentheses.

q	$\pi_0(C_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		(IQHE)	
1	0			

Above: insulators without time-reversal symmetry (i.e., systems with Q symmetry only) are classified using complex K-theory.

Right: superconductors/superfluids (systems with no symmetry or T-symmetry only) and time-reversal invariant insulators (systems with both T and Q) are classified using real K-theory.

q	$\pi_0(R_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		no symmetry (p_x+ip_y, e.g., SrRu)	T only (^{3}He-B)
1	$\mathbb{Z}_2$	no symmetry (Majorana chain)	T only ($(p_x+ip_y)\uparrow+(p_x-ip_y)\downarrow$)	T and Q (BiSb)
2	$\mathbb{Z}_2$	T only ($(\text{TMTSF})_2\text{X}$)	T and Q (HgTe)	
3	0	T and Q		
4	$\mathbb{Z}$			
5	0			
6	0			
7	0			no symmetry

Kitaev '09: Based on K-theory, Bott periodicity.
Q: Are interesting features robust/model-independent?

Related: Freed–Moore '13: Twisted equivariant K-theory classification of gapped free-fermion phases.

Classification principles

Wanted: some object (group?) classifying *gapped* free-fermion phases compatible with certain given symmetry G.

Existing literature: Differ on many basic definitions!

Basic classification principles:

- Symmetries of dynamics can preserve/reverse time/charge.
- **P**rojective **U**nitary-**A**ntiunitary symmetries $\sim$ Wigner.
- *Charged* fermionic dynamics, as opposed to *neutral* dynamics.
- Insensitive to "deformations" preserving gap.

Strategy: encode symmetry data in a C^*-algebra $\mathcal{A}$.

- "Topological" invariants are those of $\mathcal{A}$.

Charged gapped free-fermion dynamics

Non-degenerate ("gapped") dynamics

Unitary time evolution $U_t = e^{\mathrm{i}tH}$ on $(\mathscr{H}, h)$, with 0 in gap of $\mathrm{spec}(H)$. Define $\Gamma := \mathrm{sgn}(H)$, so that $U_t = e^{\mathrm{i}tH} = e^{\mathrm{i}\Gamma t|H|}$.

- Γ splits $\mathscr{H}$ into positive/negative energy subspaces. Important for *positive energy (second) quantization*[1].
- Kitaev '09: consider all H with same flattening Γ to be "homotopy" equivalent. Only grading Γ is important.

[1] Dereziński–Gérard '10, '13.

Time and charge reversing symmetries

Symmetry group $G \to \mathscr{B}_{\mathbb{R}}(\mathscr{H})$. Extra data:

- Homomorphisms[2] $\phi, \tau : G \to \{\pm 1\}$ encode whether rep. $\mathrm{g} \equiv \theta_g$ is unitary/antiunitary and time preserving/reversing:
  $\mathrm{g}\mathrm{i} = \phi(g)\mathrm{i}\mathrm{g}, \quad \mathrm{g}U_t = U_{\tau(g)t}\mathrm{g}.$
- Consequence: g is even/odd according to $c := \phi \circ \tau$,
  $\mathrm{g}\Gamma = c(g)\Gamma\mathrm{g}.$
- 2-cocycle $\sigma : G \times G \to \mathbb{T}$ encodes $\theta_{g_1}\theta_{g_2} = \sigma(g_1, g_2)\theta_{g_1 g_2}$.
- Summary: Symmetry data is (G, c, ϕ, σ), acting projectively on graded Hilbert space $(\mathscr{H}, \Gamma)$ as even/odd (anti)unitary operators according to c, ϕ —— "Graded PUA-rep"

[2] c.f. Freed–Moore '13.

Symmetry algebra: twisted crossed products

Graded PUA-rep of $(G, c, \phi, \sigma) \overset{1-1}{\longleftrightarrow}$ non-degenerate $*$-rep of associated graded *twisted crossed product C^*-algebra*[3] $\mathcal{A} := \mathbb{C} \rtimes_{(\alpha,\sigma)} G$.

- $\mathbb{C}$ is regarded as a *real* algebra.
- $\phi(g) = -1 \iff \alpha(g)(\lambda) = \bar{\lambda}$, twisted by σ.
- c determines $\mathbb{Z}_2$-grading on $\mathcal{A}$.
- All symmetry data is in $\mathcal{A}$.

Notation: $\mathbb{C} \rtimes_{(1,1)} G \longrightarrow \mathbb{C} \rtimes G$.

[3] Leptin '65, Busby '70, Packer–Raeburn '89

Example: CT-symmetries, Clifford algebras, tenfold way

Let T = "**T**ime-reversal", C = "**C**harge-conjugation". Consider $G = P \subset \{1, T\} \times \{1, C\}$ = "CT"-group.

- c, ϕ are standard, e.g., $\phi(T) = -1$, $c(C) = -1$.
- σ can be standardised using U(1) phase freedom.
- Ten possible "CT-classes"; each symmetry algebra $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} P$ is a Clifford algebra.

Example: CT-symmetries, Clifford algebras, tenfold way[4]

Generators of P	C^2	T^2	Associated algebra	Ungraded Clifford algebra	Graded Morita class
T		$+1$	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	$Cl_{1,2}$	$Cl_{0,0}$
C, T	-1	$+1$	$M_4(\mathbb{R})$	$Cl_{2,2}$	$Cl_{1,0}$
C	-1		$M_2(\mathbb{C})$	$Cl_{2,1}$	$Cl_{2,0}$
C, T	-1	-1	$M_2(\mathbb{H})$	$Cl_{3,1}$	$Cl_{3,0}$
T		-1	$\mathbb{H} \oplus \mathbb{H}$	$Cl_{3,0}$	$Cl_{4,0}$
C, T	$+1$	-1	$M_2(\mathbb{H})$	$Cl_{0,4}$	$Cl_{5,0}$
C	$+1$		$M_2(\mathbb{C})$	$Cl_{0,3}$	$Cl_{6,0}$
C, T	$+1$	$+1$	$M_4(\mathbb{R})$	$Cl_{1,3}$	$Cl_{7,0}$
N/A	N/A		$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}l_1$	$\mathbb{C}l_0$
S	$S^2 = +1$		$M_2(\mathbb{C})$	$\mathbb{C}l_2$	$\mathbb{C}l_1$

Table: The ten classes CT-symmetries (P, σ), and their corresponding Clifford-symmetry-algebras.

[4] Dyson '62, Altland–Zirnbauer–Heinzner–Huckleberry '97, '05, Abramovici–Kalugin '11 etc.

Towards K-theory groups of symmetry algebras

What are physically relevant groups for $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$?

Start with $c \equiv 1$ (trivial grading) case.

- For compact G: Representation ring/group $\mathcal{R}(G) \cong K_0(\mathcal{A})$.
- For general (G, ϕ, σ), *define* $\mathcal{R}(G, \phi, \sigma)$ to be $K_0(\mathcal{A})$.

E.g commutative case: $\mathbb{C} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)$-module $\xleftrightarrow{\text{Serre}-\text{Swan}}$ $\Gamma(E \to \mathbb{T}^d)$; reminiscent of Bloch theory and band insulators.

Q:What is "$K_0(\mathcal{A})$" for *graded* symmetry algebras ($c \not\equiv 1$)?

A1: "Super-rep group", super-Brauer group, super-division algebras. . . recovers $d = 0$ in Table.
A2: Use a model for K-theory in Karoubi '78, roughly: stable homotopy classes of grading operators compatible with $\mathcal{A}$.

A model for K-theory: Difference-groups

Consider the set $\mathrm{Grad}_{\mathcal{A}}(W)$ of possible grading operators on an *ungraded* $\mathcal{A}$-module W.

- "symmetry-compatible gapped Hamiltonians on W".

Note: $\pi_0\left(\mathrm{Grad}_{\mathcal{A}}(W)\right)$ has no group structure yet!
$\Rightarrow$ Study *differences* of compatible Hamiltonians.

- Triple (W, Γ_1, Γ_2) represents ordered difference.
- Triple is *trivial* if Γ_1, Γ_2 are homotopic within $\mathrm{Grad}_{\mathcal{A}}(W)$.
- $\oplus$ gives monoid structure to the set $\mathrm{Grad}_{\mathcal{A}}$ of all triples; trivial triples form submonoid $\mathrm{Grad}_{\mathcal{A}}^{\mathrm{triv}}$.

The *difference-group* of symmetry-compatible gapped Hamiltonians, $\mathbf{K}_0(\mathcal{A})$, is $\mathrm{Grad}_{\mathcal{A}}/\sim_{\mathrm{Grad}_{\mathcal{A}}^{\mathrm{triv}}}$.

A model for K-theory: Difference-groups

Nice properties of $\mathbf{K}_0(\cdot)$:

- $\mathbf{K}_0(\mathcal{A})$ is an abelian group, with $[W, \Gamma_1, \Gamma_2] = -[W, \Gamma_2, \Gamma_1]$.
- Path independence: $[W, \Gamma_1, \Gamma_2] + [W, \Gamma_2, \Gamma_3] = [W, \Gamma_1, \Gamma_3]$.
- $[W, \Gamma_1, \Gamma_2]$ depends only on the homotopy class of Γ_i.

Special case: for purely-even $\mathcal{A}^{\mathrm{ev}}$, our $\mathbf{K}_0(\mathcal{A}^{\mathrm{ev}})$ is one of Karoubi's models for the ordinary $K_0(\mathcal{A}^{\mathrm{ev}})$.

Karoubi '78, '08: Clifford "suspension" $\mathcal{A} \mapsto \mathcal{A}\hat{\otimes}Cl_{0,1}$ is compatible with the usual suspension $\mathcal{A} \mapsto C_0(\mathbb{R}, \mathcal{A})$, i.e.,

$$\begin{aligned} \mathbf{K}_0(\mathcal{A}\hat{\otimes}Cl_{0,n}) &\cong \mathbf{K}_0(C_0(\mathbb{R}^n, \mathcal{A})). \\ &\cong K_n(\mathcal{A}), \text{ if } \mathcal{A} = \mathcal{A}^{\mathrm{ev}}. \end{aligned}$$

Dimension shifts in Periodic Table

Common claim: Classification in d dimensions is the same as $d = 0$ classification, except for shift in by d. To what extent is this true?

- $G = G_0 \times P$, plus *mild* assumptions, symmetry algebra $\mathcal{A} \cong \mathcal{A}^{\mathrm{ev}}_{\mathbb{R}} \hat{\otimes} Cl_{r,s}$. Thus,

$$\mathbf{K}_0(\mathcal{A}) \cong K_{s-r}(\mathcal{A}^{\mathrm{ev}}_{\mathbb{R}}) \text{ ``}\cong\text{''} KR^{r-s}(X).$$

Suppose $\tilde{G} = \tilde{G}_0 \times P$ where $\tilde{G}_0$ is an extension of G_0 by $\mathbb{R}^d$. Then $\tilde{\mathcal{A}} = \mathbb{C} \rtimes_{(\tilde{\alpha},\tilde{\sigma})} \tilde{G} \cong (\mathcal{A}^{\mathrm{ev}}_{\mathbb{R}} \rtimes_{(\beta,\nu)} \mathbb{R}^d) \hat{\otimes} Cl_{r,s}$.

Q: How is $\mathbf{K}_0(\tilde{\mathcal{A}})$ related to $\mathbf{K}_0(\mathcal{A})$?

Dimension shifts in Periodic Table

Powerful results from K-theory of crossed products:

Connes–Thom isomorphism, Connes '81

$K_n(\mathcal{A} \rtimes_{(\alpha,1)} \mathbb{R}) \cong K_{n-1}(\mathcal{A})$ for any action of $\mathbb{R}$.

Packer–Raeburn stabilisation trick, Packer–Raeburn '89

Twisted crossed products can be untwisted after stabilisation:
$(\mathcal{A} \rtimes_{(\alpha,\sigma)} G) \otimes \mathcal{K} \cong (\mathcal{A} \otimes \mathcal{K}) \rtimes_{(\alpha',1)} G$.

Corollary: Dimension shifts

$K_n(\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^d) \cong K_{n-d}(\mathcal{A})$.

Dimension shifts in Periodic Table

Thus, extra $\mathbb{R}^d$ symmetry shifts degree of the difference group:

$$\mathbf{K}_0(\tilde{\mathcal{A}}) \cong K_{s-r}(\mathcal{A}_{\mathbb{R}}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d) \cong K_{s-r-d}(\mathcal{A}_{\mathbb{R}}^{\mathrm{ev}}).$$

- Note: result does not depend on how $\mathbb{R}^d$ fits in $1 \to G_0 \to \tilde{G}_0 \to \mathbb{R}^d \to 1$.
- Extra $\mathbb{R}^d$ symmetries may be *projectively* realised (IQHE).
- Some extra assumptions needed for discretised version of this result.

Periodic Table of difference-groups of gapped topological phases

n	C^2	T^2	$\mathbf{K}_0(\mathcal{A}) \cong K_{n-d}(\mathbb{R})$ or $K_{n-d}(\mathbb{C})$ $d=0$	$d=1$	$d=2$	$d=3$
0		$+1$	$\mathbb{Z}$	0	0	0
1	$+1$	$+1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
2	$+1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
3	$+1$	-1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
4		-1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
5	-1	-1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
6	-1		0	0	$\mathbb{Z}$	0
7	-1	$+1$	0	0	0	$\mathbb{Z}$
0	N/A		$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	$S^2 = +1$		0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table: Vertical degree shifts — effect on $\mathbf{K}_0(\mathcal{A})$ of tensoring with a Clifford algebra. Horizontal shifts — Connes–Thom isomorphism. Twofold and eightfold periodicities — Bott periodicity. Assuming translational symmetry $\times$ CT-symmetry.

General remarks

- Conceptual advantage: all symmetries are treated on an equal footing. These include T, C, projective symmetries (e.g. IQHE), $\mathbb{Z}^d$ (band insulators), and extra spatial translations $\mathbb{R}^d$.
- Phenomenon of "dimension shift" is robust and model-independent.
- We see why T-symmetry needs to be broken for IQHE, but $\mathbb{Z}_2$-invariant possible for QSHE.
- Not restricted to condensed matter applications.