Different ways to geometrize physics FFP 2014

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Content:

There are three kinds of relationships between physics and geometry. The first one is some kind of active identifying :
1-Euclidean geometry and mechanics, cartesian geometry and physics, Newtonian dynamics The role the philosophy plays is specific (Cartesius, Leibniz, Kant).
The second one is an interactive game under domination of one kind of geometry : 2-differential, complex, algebraic geometries and their mixing. With respect to the first one we can see in this role an reflexivity with one degree more: the nature of mathematical object, complex, differential, or algebraic is taken into account.

The third one consists of the building of a mathematical reflexive structure that try to take into account a vision on the mathematical corpus and its unity form and at the same time its possibility to accommodate and to think about physics through its conceptual structure, 3- I mean the category theory and its branch : topos theory

My first point : the idea that mathematics is particularly well suited to the knowledge of the physical reality and capable of expressing properties has going for historical reasons. The most important fact in the history of science by its duration and its place was **the coupling between Newtonian mechanics and Euclidean geometry**.

The building has cracked in the XIXth century by a transition to another form of abstraction and reflexivity at work in mathematics. Could we describe the relation between the physical reality and the Euclidean geometry? I cannot discuss the question of whether Euclidean geometry was suggested by the perceptual datum. (cognitive sciences)

It should be clear that if one wants to account for the shape of three-dimensional Euclidean figures must be sought in the requirements of regularity, simplicity, spacings, orthogonality and not perceptive suggestions. Plane geometry it is an abstraction compared to the field of solid fields, The answer as for the Euclidean geometry and its duration and adaptability is given by E. Cartan in the 20th century. Euclidean space is the only one who admits an orthogonal triple system consisting of totally geodesic surfaces. It presents itself as a space without torsion and with constant curvature zero.

It is all the more degenerated areas and has the largest group of automorphisms and he serves as a reference for others. In a variety of constant curvature metrics reports about a point are the same as around any other. can move from one region to another by a positive isometry group of displacements (rotations translations). what characterizes Euclidean geometry is the group of displacements which leave invariant the figures in this geometry.

In fact it can be showed that the geometrical abstraction is possible because the Euclidean object is invariant under the displacements, so the Euclidean geometry results from a determined elaboration.

Coupling with rational mechanics is to add the local motion (assumed continuous physical process) objects to a theory that does not undergo any change in this movement. The complex: geometry-rational mechanics is a dual theoretical movement.

The analysis of this double process by which the physical phenomenon is mathematized at the same time as the implied(involved) mathematical idealities are developed show the intricacy of the geometry and the rational mechanics from the beginning of this one.

The reality to which the mathematics are supposed to apply is already established(constituted) in a way *a priori* by the determinations which the Euclidian geometry imposes when the study of the movement is beginning. Its structure (of EG) is pre-ordered to welcome it.

The magic of complex numbers

Magic? It is the worlds Penrose uses to characterize the strengh of complex geometry for physics. Strong understanding of physical but very specific, beyond the (classical) rationality Complex system number we find to be so fundamental to the operations of QM as opposed to the real number system which had provided the foundation of all successful previous theories.

The importance of complex numbers or more specifically, the importance of holomorphicity (or complex analyticity)in the basis of physics is indeed, Penrose says, to be viewed as a 'natural' thing. Nature itself appears to harness this magic in weaving her universe at its deepest levels. And Penrose questions wether this is really a true structure of our world, or wether it is merely the mathematical utility of these numbers that had led to their extensive use in physical theory.

There is a complex structure. There is no complex object. And it is clear that this complex structure allows to conceptualize the framework in QT, underlying, as they do, the fundamental quantum superposition principle, the Schrödinger equation, the infinite -dimensional 'complex structure' that comes about in QFT.

Complex

reality is a structuralist one. There exist several levels of this. They are some properties of the complex frame of the complex geometry that allow to conceive some characteristics of the physical conceptual reality. Is this conceptual reality an ontological one? All physical entities take place inside this complex background. The complex numbers and their properties are, so to say, ontological coordinates of the physical entities we have to deal with.

Complexification

Let us consider this processing. It doesn' t mean that one reduces in some way physical reality to the complex geometrical one. It means that we have to conceive the physical reality by means this complex background: all geometrical objects: sphere, space (Minkowski), Hilbert space are complex. Is the processing of complexification a constructivist one? The question is wether we have to be constructivist realist?

I would take the Riemann sphere as an example. We construct the Riemann sphere. We regard the sphere to be constructed from two 'coordinate patches', one of which is the z-plane and the other the w-plane. All but two points of the sphere are assigned both a z-coordinate and a w-coordinate (related by the Mœbius transformation). A point has only a z coordinate and another a w-coordinate. We use either z or w or both in order to define conformal structure. The Riemann sphere so defined is rather an abstract entity, but this abstract form allows a purely conceptual approach to the physical reality.

This approach is geometrical, metaphorical, and conceptual at the same time. It exploits the resources of the sphere from the three points of view.

Penrose can explain in this way the celestial sphere."If you look up at the sky on a clear cloudless light , you appear to see a hemispherical dome about you punctuated by myriads of stars". "In fact you are realistically picturing the family of light ray that constitute the light cone centered at the event O that is occupied by our eye at the moment that you perceive the celestial scene".

Complex number structure is indeed fundamental for QM also. a) The 'amplitudes' that appear as coefficients in the superposition law of QM are complex numbers, leading to the complex Hilbert space. b) the amplitudes play basic roles in providing probabilities, when a measurement takes place c) there is, Penrose says, a strong interconnection between these complex numbers and spatial geometry.

The quantum mechanics of a spin 1/2 particle, the possible states of spin correspond to the different spatial directions via the notion of a *Riemann sphere*. Penrose recalls that spin states for higher spin can also be described in terms of the spatial geometry of the Riemann sphere by means of the Majorana representation

Realism of structure?

The Riemann sphere is a particular construal. From this point of view, one can accept to be a constructivist structuralist. But from the other side this construct holds for very different physics characteristics. The same is true for complex numbers, (Riemann sphere is a complex object). For this reason the construal is only partial. Complex geometry, with particular objects, play also the role of a structural background.

There is in fact a disappearance of the classical notion of object. What is exactly the sense of this non locality in a structuralist view? This essential characteristic means that the structure as reconceptualizing of object is also non local. At first glance the meaning of this characteristic is negative. And it would be possible to speak on a non local object. Comment and see the third part

**The question is: is the locality connected with the notion of an object? No necessarily. We must distinguish between the notion of support for an conceptual construction and its space-time localization. If we reject this distinction the non locality forces us to reject in the same gesture the notion of object. If not we have the possibility to remain at level of the Kantian categories without the sensible intuition.*

In Penrose view, and this is a very important conception, ordinary space time notions are not initially among the ingredient of the twistor theory but are to be *constructed* from them. Twistors are not situated in space and time. Even these ones are not 'realized' in space and time. And the twistor description of space time turns out to be a non-local one, There is a fundamentally 'holistic' character to the twistor description of physical fields that comes about via a remarkable feature of complex magic. The non local aspects of twistor theory are intimately bound up with the most important of its underlying motivations,"namely the desire to exploit the magic of complex number in a belief that Nature herself may well be dependent upon such things at a deep level".

Twistor space, reversal point of view

The first step towards the understanding of twistor ideas is to think of a twistor as a representing a *light ray* in ordinary space time. Such a light ray is providing the primitive ' causal link' between a pair of *events*. But events are themselves to be regarded as ısecondary constructs these being obtained from their roles in as intersection of light rays. An event **R** may be characterized by means of the family of light ray that pass through **R**.
In the normal spacetime picture a light ray **Z** is a locus and an event **R** is a point
in the twistor space light ray is described as a *point* Z, and an event is described as a *locus* R.
Striking reversal.

The twistor space, whose individual points represent light rays in **M** is denoted by **PN**. The point **Z** in **PN** correspond to the locus $\mathcal{Z}$ inside **M** and the point $\mathcal{R}$ in **M** correspond to the locus **R** (a Riemann sphere) inside **PN**. "An essential part of the philosophy of twistor theory is that ordinary physical notions, which are described in spacetime terms are to be translated into an equivalent (but non-locally related) description in twistor space"

** This locus **R** inside **PN** describes, as we saw, the 'celestial sphere', of an observer at $\mathcal{R}$
This sphere is naturally a Riemann sphere which is a complex 1-dimensional space (a complex curve). We think of spacetime points as holomorphic objects in the twistor space **PN** "in accordance with the complex-number philosophy underlying twistor theory." What does it means in the frame of the realist philosophy? Objects are holomorphic objects. We have seen that points are relational objects. Single point in **M**. It is the same for the Riemann sphere. But it inherits all properties of holomorphicity. And these properties capture the concepts that are basic in the Penrose view of physics.

PN is not a complex manifold because it has only five real dimensions and five is an odd number. Penrose proposes we make 'our light rays a little more like physical massless particles, by assigning them both spin (helicity) and energy. Then we get a *six* dimensional space Penrose calls **PT**. The space **PN** sits inside **PT**, dividing it into two complex manifold pieces $\mathbf{PT}^{+}$ and $\mathbf{PT}^{-}$, where $\mathbf{PT}^{+}$ (resp. $\mathbf{PT}^{-}$) may be thought of as representing massless particles of positive (resp. negative) helicity .

Ontological analysis

Penrose explains that twistors are not to be thought of *as* massless particles. Instead, twistors provide variables in terms of which massless particles are to be expressed. It is doubtless a realist proposition. Massless particle is more than its expression in terms of twistor.

We get by means of Penrose view what we could call a quantified realism.

The appropriate 'quantization' procedures must be applied within twistor space rather than within the space time. In the conventional approach 'events' are left intact whereas 'null cones' becomes fuzzy, in a twistor pace approach it is the 'light ray ' that are left intact whereas 'events' become fuzzy.

*** The twistorial perspective transform the notion itself of the existence of physical reality: the twistor space (**PN**) retains some kind of existence (there would still be light rays), but the condition of their intersection would become subject to quantum uncertainties. Accordingly, Penrose says, the notion of 'space time point' would instead become 'fuzzy'.

For the sake of briefness, it is a realism of compactification.
Property is transformed. Compactified complexified reality.
twistor sheaf cohomology
Homogenous functions of single twistors generate solutions of the massless field equations was a miracle: magic? Realism : there is more in mathematics (platonism) no locality e.g. And very important the notion of property is transformed.

This kind of geometrization consists of identifying physical reality with a mathematical, geometrical one: entities of complex geometry. It would be possible to follow farther this complex geometrisation.

At the same epistemological level would be differential geometry (or algebraic). So is the case of Riemannian geometry

A topos is a category with extra properties. These extra properties make a topos"looks like"**Sets**: many mathematical operations which can be done in set theory can be done in a general topos . The question is what for essential operation make possible the development of a geometrical or generally a mathematical theory? The question was treated through set theory.

I would like to present you the main features of a very developed theory which has occurred in the conditions of theoretical problems of mathematical physics. The use of topos theory in the foundations of physics and u in particular the foundations of quantum theory was suggested by Chris Isham more than 15 years ago. Subsequently these ideas were developed in application to the Kochen-Specker theorem with Jeremy Butterfield.

Cecilia Flori in hers book"A first course in Topos Quantum Theory"says that a great deal of effort in fundamental physics is spent on an elusive theory of quantum gravity which an attempt to combine theory of special and general relativity with quantum theory, various attempt to formulate such a theory of quantum gravity has been made, but none has fully succeeded in becoming *the* quantum quantum theory of gravity. One possibility of the failure for reaching an agreement on a theory of quantum gravity might be the presence of unresolved fundamental issues already present in quantum theory, first pointed out by John Bell.

The hidden -variables interpretation which assumes that the formalism of QT is incomplete. In particular they postulate the existence of a set of hidden variables which belong to some space. Such a set of hidden variables together with the state ψ of a system allow us to uniquely determine the value of all quantities of that system. But the majority of physicists adhere to the "standard" Copenhagen interpretation and do not regarded it as neither incomplete nor problematic.

Which are the main conceptual issues in QT? How can these issues be solved within a new theoretical framework of QT?

The ideas put forward by C. Isham, A. Döring, J. Butterfield and others have proposed that the main issues in the standard quantum formalism are CF (A) the use of critical mathematical ingredients which seem to assume certain properties of space and /or time which are not entirely justified.(such a priori assumptions not compatible with a theory of QG) (B) the instrumental interpretation of QT that denies the possibility of talking about without any reference to an external observer.

Through a reformulation of QT in terms of a different mathematical framework called topos. The reason for choosing topos theory is that it 'looks like' set and is equipped with an internal logic. These features will allow for a reformulation of QT, which is more realist and which does not rest on *a priori* assumptions about the nature of space and time.
Remark on the meta level as a way to get out the space and time

The adoption for a more realist interpretation should be considered only as a possible strategy to overcome the instrumental interpretation of QT and to make contact with the interpretation of classical physics but not as a universal criterion for a more acceptable theory. A more 'general' realist interpretation of QT which can be reduced to a realist interpretation in the limit applicable to classical physics. It is a general phenomenon some interpretation of a new theory is at same time a new one of the classical previous theory and this is realized by means of a new mathematical framework

Analogy and meta analogy

The main idea in the topos formulation of normal QT is that using topos theory to redefine the mathematical structure of QT leads to a reformulation of QT in such a way that it is made to 'look like' classical physics.
a) no fundamental role is played by the continuum b) propositions can be given truth value without needing to invoke the concepts of 'measurement' or 'observer'.

Why Isham and Döring searched for a reformulation of a quantum theory that is more realist than the existing one? The reasons for that are the following

- As it stands QT is non-realist. The Kochen-Specker theorem implies that any statement regarding state of affairs formulated within the theory, acquires meaning counterfactually. It is hard to avoid the Copenhagen interpretation
- Notion of 'measurement and 'external observer' pose problems when dealing with cosmology
- The existence of the Planck scale suggests that there is no *a priori* justification for the adoption of the notion of a continuum in QT

In this case, the intervention of the category theory (topos) is directly linked with a philosophical intention: it is possible because of the level at which the theory is located: It has the ability to reproduce in theory a position overlooking a form of realism

What is the underlying structure which makes classical physics a realist theory?

1- The existence of a state space S

2- Physical quantities are represented by functions from the state space to the reals. Thus each physical quantity A, is represented by a function

$$f_A : S \to \mathbb{R}$$

3- Propositions (Isham, Döring, Fiori) are of the form " $A \in \Delta$ The value of the quantity A lies in the subset $\Delta \subseteq \mathbb{R}$. There are represented by subsets of the state space S : those subspaces for which the proposition is true. For example for the proposition "$A \in \Delta$", this is just

$$f_A^{-1}(\Delta) = \{s \in S | f_A(s) \in \Delta\}$$

The collection of all such subsets forms a Boolean algebra *Sub*(S).

States ψ are identified with Boolean-algebra homomorphims

$$\psi : Sub(S) \rightarrow \{0, 1\}$$

from the Boolean algebra *SubS* to the two elements $\{0, 1\}$. Here 0 and 1 can be identified as 'false' and 'true' respectively. The identification of states with such maps follows from identifying propositions with subsets of S. Indeed to each subset $f_A^{-1}((\ \Delta\})$ there is associated a characteristic function $\chi_{A \in \Delta} : S \rightarrow \{0, 1\} \subset \mathbb{R}$

Thus each state s either lies in $f_A^{-1}((\ \Delta\})$ or it does not. Equivalently, given a state s every proposition about the values of physical quantities in that state is either true or false. Here we have the minimal form of a physical theory

quantum analogues of these requirements

To consider the appropriate mathematical framework in which to reformulate the theory. The choice fell in topos theory. The paramount reason is that in any topos distributive logic arise in a natural way : a topos has an internal logical structure that is similar , in many ways, to the way in which Boolean algebra arises in set theory. the sub-objects of our state should form some sort of logical algebra

Isham et al. achieved to identify which topos is the right one by noticing that the possibility of obtaining a 'neo-realist' reformulation of QT lied in the idea of *context*. because of the KS theorem the only way of obtaining quantum analogues of requirements 1, 2, 3, and 4 is by defining them with respect to commutative sub-algebras (the contexts) of the "bon"-commuting algebras $\mathcal{B}(\mathcal{H})$ of all bounded operators on the quantum theory's of Hilbert space.

Thus 'locally' with respect to these contexts QT effectively behaves classically. The idea is to try and define each quantum object locally in terms of these abelian contexts. The key feature (all this in Cecilia Fiori and Isham) is that the collections of all these *contexts* or *classical snapshots* form a category ordered by inclusion. As it is noticed by CF although one defines each quantum object locally, the global information is never lost, since it is put back into the picture by the categorical structure of the collection of all these classical snapshots

This form of a geometrization is like a transcendental one: it provides an *a priori* structure within which it is possible to think on the relation between local and global as form of the relation between classical and quantum theory

The task is to find a topos which allows you to define a quantum object as (roughly speaking (CF)) a collection of classical approximations This can be done through the topos of pre sheaves over the category of abelian sub-algebras. In terms of this topos of pre sheaves, QT can be redefined so that it retains some realism and its interpretation will be more easy

Philosophically the main reason for such thought processes is to analyze how the mathematical formalism of QT leads to a non-realist interpretation of the theory and then to show that topos QT is a way of overcoming conceptual problems of the non-realist interpretation. The QT will be redefined in the novel language of topos theory

At this level this view is a reflexive one. Reflection focuses on the theory and the relationship of theory to previous at the same time through the dichotomy local /global and approximation of quantum using the classical .

I will follow the questions CF asks concerning Theory of Physics. A theory of physics can be seen as a mathematical model which tries to answer three of fundamental questions humanity has been and still is struggling to answer:
1- What is a thing? (Heidegger)
2- How are "things" related to one another?
3- How to get to know (1) and (2) ?

The first question is related to ontological and metaphysical issues. This is related to the manner the metaphysics that what a thing consists in. Beyond the Kantian philosophy. The answer Heidegger gives is an ontological one (distinction between Etre and Etant). The two other questions are more epistemological

Understand the physics comes down to understanding the constraints exercised on us the answers to these questions.

The two mains physics theories which presupposes to answer the above question are

- 1 Classical physics
- 2 Quantum theory

The way in which these two theories have answered the above question is by defining a mathematical model which supposed to describe nature. May be only to give a setting for describing nature. The interpretation of this mathematical model then, in turn, gives rise to a philosophical view of the world which provides an answer to the questions. In classical physics the mathematical model developed is in accordance with our common believes about the world. This is to be discussed. (Our common believes modeled classical theory) classical theory is relative to our initial objective.

Any theory of physics should address the following issues:
1- What is the system under investigation?
2- What is the ontological status of physical terms?
3- What is the epistemological status of physical terms?
4- How physical statements can be verified or falsified?
5- What is the relation between the mathematical model and the physical world?
The mathematical tools used to describe a physical system encode a philosophical position regarding the world. (Inside and outside).
2 and 3 are difficult.

Philosophical Position of Classical Theory

A classical theory is an realist one in the sense that
(i) properties can be ascribed to a system at any given time and do not depend on the act of measuring
(ii) the underlying logic is Boolean (classical) logic which the same logic we employ in our language. this is to be discussed. Roughly speaking what we can say is that it the common view and even philosophical view on the every day language.

Concepts in classical theory.
1- State space S collection of all states $s_i \in S$ each s_i at a given time t_i embodies all the properties of the system at that time
2- physical quantities identified with the collection of values it can have for a given system.
3- propositions these are of the form " $A \in \Lambda$", meaning the physical quantity A takes its value in the interval $\Lambda \in \mathbb{R}$. In classical physics such a proposition is identified with the collection of states $s_i \in S$ for which the quantity A does have value which lie in the interval Λ

Behind Quantum theory

The theory is non-realist. the above conditions (properties do not depend on the act of measuring, the underlying is Boolean) do not strictly in QT. The distinction between measuring apparatus and and measuring system makes that the act of measuring gets ascribed a special status. Assigning a probabilistic spread of outcomes.The very concept of properties ceases to have its common sense meaning. (logical consequences, cf quantum logic) It is as if properties acquire the status of latent attributes which are brought into existence by the act of measuring (Cecilia F). It becomes meaningless to talk about a physical system as possessing properties

The interpretation that results is the so called the instrumentalist interpretation In this view : there is no distinction between ontological and epistemological status of physical terms. Very roughly,one can say that QT is a mathematical model which, in a way gives rise to "things" through the measurement process. Thus a thing becomes simply a result of a measurement which only describes what we assume exists "out there". In turn, physical statements seem only represent our knowledge of events rather than events themselves.

Mixing of philosophy and mathematics

First occurrence. Topos quantum theory can be seen as a contextual quantum theory in the sense of each element is defined as a collection of 'context dependent' description. Such context dependent description will turn out to be classical snapshots. The above mentioned contexts are abelian von Neumann sub-algebras the collection of which forms a category under sub-algebra inclusion which we denote as $\mathcal{V}(\mathcal{H})$. Although locally quantum theory can be defined in term of local classical snapshots, the global quantum /quantum information is put back into the picture by the categorical structure of the collection of all such classical snapshots.

A model is defined to be strongly contextual if its support has no global sections: equivalently the propositional formulas defining its support are not simultaneously satisfiable

Three ideas or more philosophical concepts are associated : locality, contextuality, 'classicality'. locality. Here locality comes from the structure of sub algebra of $\mathcal{B}(\mathcal{H})$.This locality is determined by a sub algebraic structure, the Von Neumann sub-algebra. it is locality in the sense of pre sheaf, but so to speak transferred in the sub-algebra (there is no global section). In the topos approach to QT there is a type of contextuality different of the one in the context of Kochen-Specker theorem.

To summarize
(i) we first consider different contexts that represent classical snapshots
(ii) we then define our QT locally in terms of such classical snapshots; therefore in a way performing a classical approximation
(IIi) the quantum information, which is lost at the local level is however put back into the picture by the categorical structure of all the classical contexts (comment)

The category of classical snapshots we will use is the category of abelian von Neumann sub-algebras of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on the Hilbert space. Until which limits does the concept of context posses a meaning? This meaning is formalized by the concept of sub algebra of Von Neumann. It is sufficient the philosophical meaning is strictly restricted to the algebraic structure. It is neither exactly a formalization nor a meaning giving.

A category of Abelian von Neumann Sub-Algebra. We consider the algebra of bounded operators on a Hilbert space $\mathcal{B}(\mathcal{H})$, A quantum system can be represented by a von Neumann algebra $\mathcal{N}$ which is identified with a sub-algebra of $\mathcal{B}(\mathcal{H})$. A von Neumann algebra is a $*$ algebra of bounded operators.

To say exactly : the category $\mathcal{V}(\mathcal{H})$ of abelian von - Neumann sub-algebras has
1- *objects* $V \in \mathcal{V}(\mathcal{H})$ abelian von Neumann sub-algebras
2- *morphisms* given two sub algebras V_1, V_2 there exists an arrow between them $V_1 \to V_2$ iff $V_1 \subseteq V_2$

If we consider an algebra V' s t. $V' \subseteq V$ then the set of self adjoint operators present in V' which is denote by V'_{sa} will be smaller than the set of self adjoint operators in V i.e. $V'_{sa} \subseteq V_{sa}$ that means the context V' contains less physical information s. t. by viewing the system from the context V' we know less about it then when viewing from the context V. This represents a type of coarse graining which arises when going from a context with more information V to a context with less information V'.

For each context $V \in \mathcal{V}(\mathcal{H})$ Isham et ali managed to reproduce a situation analogous to classical physics in which self-adjoint operators are identified with function from a space to the reals. In this sense the topological space $\underline{\Sigma}_V$ can be interpreted as a local state space, one for each $V \in \mathcal{V}(\mathcal{H})$. The complete quantum picture is only given when we consider the collection of all the local spaces, since not all operators are contained in a single algebra. Such a collection of local spaces which define the topos analogue of the state space is called the spectral pre sheaf.

The spectral pre sheaf $\underline{\Sigma}$ is the covariant functor from the category $\mathcal{V}(\mathcal{H})^{op}$ to **Sets**(equivalently, the contravariant functor from $\mathcal{V}(\mathcal{H})$ to **Sets**)

- Objects. Given an object V in $\mathcal{V}(\mathcal{H})$ the associated set $(\underline{\Sigma}(V) := (\underline{\Sigma}_V)$ is defined to be the Gel'fand spectrum of the unital commutative von Neumann sub algebras V, i. e. the set of all multiplicative linear functionals : $\lambda : V \to \mathbb{C}$ s.t. $\lambda(\hat{\mathbf{1}}) = 1$
- Morphisms. Given a morphism $i_{VV'} : V' \to V(V' \subseteq V)$ in $\mathcal{V}(\mathcal{H})$, the associated function $\underline{\Sigma}_{VV'} := \underline{\Sigma}_V := \underline{\Sigma}(i_{VV'}) : \underline{\Sigma}_V \to \underline{\Sigma}_{V'}$ is defined for all $\lambda \in \underline{\Sigma}_V$ to be the restriction of functional $\lambda : V \to \mathbb{C}$ to the sub-algebra $V' \subseteq V$ i.e. $\underline{\Sigma}(i_{VV'}) := \lambda_{|V'}$

The concept of daseineisation

This is a interesting concept because it is a mixture of philosophical concept or category and of mathematical physics conceptual meaning. I will take the example of the use of this concept which is required to represent proposition in topos quantum theory. Literally translated it means "bring into existence". Its origin is Heidegger thesis of Dasein position as a this paradoxical Being who is confronted with the possibility of his death. He is. always in the world . It is a synonymous of existence.

That means that a theoretical, geometrical object should and must exist. This his requirement is that of mathematical physics. This is a form of Kantian schematization when considered from the ontological point of view.

As for the physics such an operation is used when trying to represent a proposition which locally defined by a projection operator, in all contexts. It could be the case that, given a context V, the proposition represented by the projection operator $\hat{P}$ does belong to V, i.e. $P \notin V$. It is in such cases that the proces of daseinisation is used. What it does is essentially to approximate the proposition $\hat{P}$ as best as possible so that it now belongs to V. This process of approximation is called *coarse graining*

How propositions are represented in $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$? They are identified with terms of type $P(\Sigma)$ i.e. with sub -objects of the state space object. In standard QT propositions are represented by projection operators, on the other hand, in topos QT they are identified with clopen (both open and closed) sub-objects of the spectral pre sheaf. A *clopen* sub-object (comment here and below) $\underline{S} \subseteq \underline{\Sigma}$is an object s. t. , for each context $V \in \mathcal{V}(\mathcal{H})$ the set $\underline{S}(V)$ is a clopen subset of $\underline{\Sigma}(V)$, the latter is equipped with usual compact Hausdorf spectral topology.

In order to understand how propositions are represented by clopen subsets of the state space we need to introduce the concept of 'daseinistaion'. What daseinisation does is to approximate operators so as to 'fit' into any given context V. The formalism defined so far is contextual then the proposition we will consider has to be studied with respect to each context $V \in \mathcal{V}(\mathcal{H})$

We would like e.g. to analyze the projection operator $\hat{P}$ which corresponds via the spectral theorem to the proposition "$A \in \Delta$". In particular let, let us take a context V s. t. $\hat{P} \notin P(V)$ (the lattice of projection operator in V). We need to define a projection operator which does belong to V and which is related in some way to our original projection operator V. This can be achieved by approximating $\hat{P}$ from above in $V$by the 'smallest' projection greater than or equal to $\hat{P}$.

More precisely the outer daseinisation $\delta^{\circ}(\hat{P})$, of $\hat{P}$ is defined for each context V by

$$\delta^{\circ}(\hat{P})_V := \bigwedge\{\hat{R} \in P(V) | \hat{R} \geq \hat{P}\}$$

Since projection operators represent propositions $\delta^{\circ}(\hat{P})_V$ is a coarse graining (remark) of the proposition "$A \in \Delta$". This process of outer daseinisation takes place for all contexts and gives for each projection operator $\hat{P}$, a collection of daseined projection operators i.e.

$$\hat{P} \mapsto \{\delta^0(\hat{P})_v | V \in \mathcal{V}(\mathcal{H})\}$$

Because of the Gel'fand transform, to each operator $\hat{P} \in P(V)$ there is associated map the map $\overline{P} : \underline{\Sigma}_V \to \mathbb{C}$ which takes values in $\{0, 1\} \subset \mathbb{R} \subset \mathbb{C}$ since $\hat{P}$ is a projection operator. Thus $\overline{P}$ is a characteristic function of the subset $S_{\hat{P}} \subseteq \underline{\Sigma}(V)$ defined by

$$S_{\hat{P}} := \{\lambda \in \underline{\Sigma}(V) | \overline{P}(\lambda) := \lambda(\hat{P}) = 1\}$$

Since $\overline{P}$ is continuous with respect to the spectral topology on $\underline{\Sigma}(V)$, then $\overline{P}^{-1} = S_{\hat{P}}$ is a clopen (comment) subset of $\underline{\Sigma}(V)$ because both $\{0\}$ and $\{1\}$ are clopen subsets of the Hausdorff space $\mathbb{C}$.

Through the Gel'fand transform it is then possible to define a bijective map between projection operators and clopen subsets of $\underline{\Sigma}_V := \underline{\Sigma}(V)$ where for each *context* V

$$S_{\delta^0(\hat{P})_V} := \{\lambda \in \underline{\Sigma}_V | \lambda(\delta^0(\hat{P})_V) = 1\}$$

This correspondence between projection operators and clopen sub-objects of the special pre sheaf $\underline{\Sigma}$, which we denote as $Sub_{cl}(\underline{\Sigma})$, implies the existence of a lattice homomorphism for each V

$$\mathfrak{G} : P(V) \rightarrow Sub_{cl}(\underline{\Sigma})_V$$

Such that

$$\delta^0(\hat{P})_V \mapsto \mathfrak{G}(\delta^0(\hat{P})_V) := S_{\delta^0(\hat{P})_V}$$

Here $Sub_{cl}(\underline{\Sigma})$ is the lattice of clopen subsets of the spectrum $\underline{\Sigma}_V$ with lattice operations given by intersection and union, while the lattice ordering is given by subset inclusion. It can be shown that the collection of subsets $\{S_{\delta^0(\hat{P})_V}\}$, $V \in \mathcal{V}(\mathcal{H})$ induces a sub-object of $\underline{\Sigma}(commentofsubobject)$

We know that the sub-objects of any object in a topos form a Heyting algebra. The sub-objects of of the spectral presheaf form a Heyting algebra. However we are only considering the clopen sub-objects . The collection of all *clopen* sub-objects of $\underline{\Sigma}$ forms a Heyting algebra it is a theorem.

To summarize Topos as topos of structural pre sheaf : category of all sub-algebras and to developp consequences of this building. All physical elements of quantum theory are translated at this level. Therefore daseinisation. and analogue of space of state, interpretation of physical quantities, and then for probabilities . Comparaison. Euclidean, Penrosean, toposic