Model Building in Almost-Commutative Geometry

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Overview

1. Basic Ideas
2. Geometry
3. Physics
4. Beyond the Standard Model
5. Conclusions

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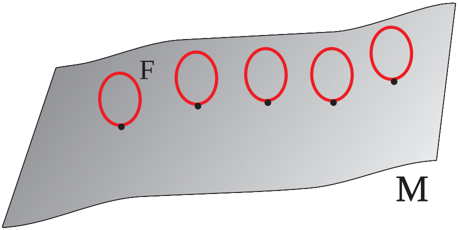
Analogy: Almost-comm. geometry $\leftrightarrow$ Kaluza-Klein space

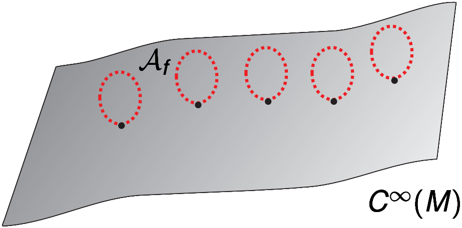


Idea:

$M \rightarrow C^\infty(M)$, $F \rightarrow$ some "finite space",

differential geometry $\rightarrow$ spectral triple

Almost-commutative Geometry

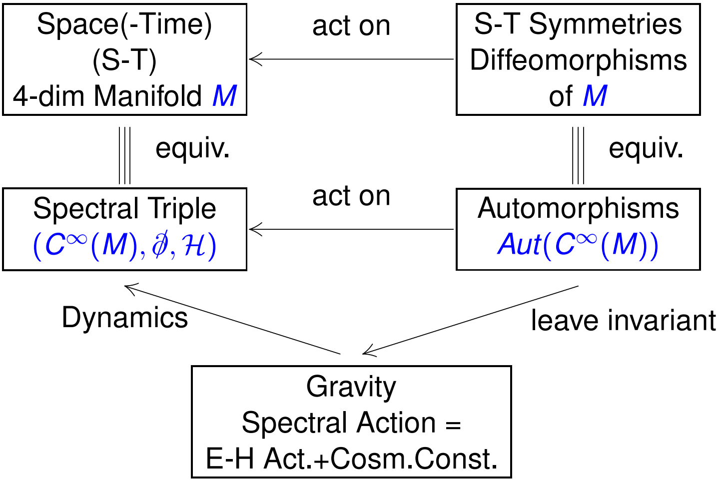


Replacing manifolds by algebras

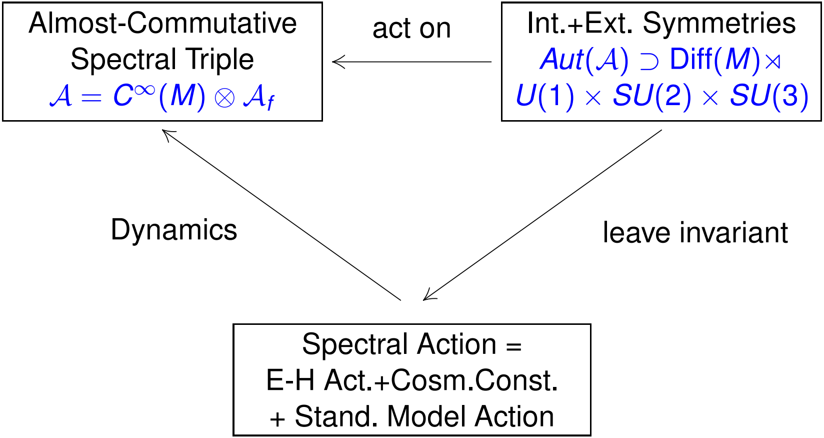
extra dimension: $F \to \mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$

Kaluza-Klein space: $M \times F \to \mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$

Euclidean space(-time)!



Almost-Commutative Standard Model (A.Chamseddine, A.Connes):



The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The generalised Dirac operator plays a multiple role:

$$D = \partial\!\!\!/ \otimes 1_f + \psi\!\!\!/ + \gamma_5 \otimes \Phi$$

Higgs & Gauge Bosons

Particle Dynamics, Ferm. Mass Matrix

Metric of M, Internal Metric

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An even, real spectral triple $(\mathcal{A}, \mathcal{H}, D)$

The ingredients (A. Connes):

- A real, associative, unital pre-C^*-algebra $\mathcal{A}$
- A Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is faithfully represented via a representation ρ
- A self-adjoint operator D with compact resolvent, the Dirac operator
- An anti-unitary operator J on $\mathcal{H}$, the real structure or charge conjugation
- A unitary operator γ on $\mathcal{H}$, the chirality or volume element

The axioms of noncommutative geometry (A. Connes):

Axiom 1: Classical Dimension n (we assume n even)

Axiom 2: Regularity

Axiom 3: Finiteness

Axiom 4: First Order of the Dirac Operator

Axiom 5: Reality

Axiom 6: Orientability

Axiom 7: Poincaré Duality

Finite spectral triples:

- $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$
  $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$
- gauge group: $Aut^e(M_n(\mathbb{C})) = U^{nc}(M_n(\mathbb{C}))$,
- $\mathcal{H}_f \simeq \mathbb{C}^N$
  N is the total number of particles
  left-/right-handed particles/antiparticles counted separately
- $D_f \in M_N(\mathbb{C})$, D_f is the fermionic mass matrix.

Axioms $\rightarrow$ Restrictions for D_f and $\mathcal{H}_f$

Almost-commutative geometry:

An almost-commutative spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a tensor product of a spectral triple of a manifold M $(\mathcal{A}_M = C^\infty(\mathcal{M}), \mathcal{H}_M, D_M = \partial\!\!\!/)$ with dimensions $n_M > 0$ and a finite spectral triple $(\mathcal{A}_f, \mathcal{H}_f, D_f)$ with metric dimension $n_f = 0$ (i.e. $\mathcal{A}_f$ matrix algebra).

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_f, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_f,$$

$$J = J_M \otimes J_f, \quad \gamma = \gamma_5 \otimes \gamma_f,$$

$$D = \partial\!\!\!/ \otimes 1_f + \gamma_5 \otimes D_f$$

$$\mathrm{Aut}(C^\infty(\mathcal{M}) \otimes \mathcal{A}_f) \simeq \mathrm{Diff}(M) \rtimes U^{nc}(\mathcal{A}_f)$$

The geometric setup imposes constraints:

- mathematical axioms
  $\rightarrow$ Restrictions on particle content
- symmetries of finite space
  $\rightarrow$ determines gauge group
- representation of matrix algebra
  $\rightarrow$ representation of gauge group
  (only fundamental and adjoint representations)
- Dirac operator $\rightarrow$ allowed mass terms / Higgs fields

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The generalised Dirac operator

The Dirac operator $\partial\!\!\!/ \otimes 1_f + \gamma_5 \otimes D_f$ is fluctuated with inner unitaries $\mathcal{U}^{nc}(\mathcal{A}_f)$ and becomes

$$D = \partial\!\!\!/ \otimes 1_f + \phi\!\!\!/ + \gamma_5 \otimes \Phi$$

The Spectral Action (A. Connes, A. Chamseddine 1996)

$$(\Psi, D\Psi) + S_D(\Lambda) \quad \text{with } \Psi \in \mathcal{H}$$

- $(\Psi, D\Psi)$ = fermionic action
  includes Yukawa couplings
  & fermion–gauge boson interactions
- $S_D(\Lambda)$ = $\sharp$ eigenvalues of D up to cut-off Λ
  = Einstein-Hilbert action + Cosm. Const.
  + full bosonic SM action + constraints at Λ
- constraints => less free parameters than classical SM

The Standard Model

- $\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$
- $\mathcal{U}^{nc}(\mathcal{A}_f) = SU(2) \times U(3)$
- $\mathcal{H}_f = \mathcal{H}_{SM}$ Hilbert space of minimal standard model fermion multiplets
- D_f: Fermionic mass matrix with CKM matrix and PMNS matrix
  $D_f \to \Phi$ Higgs field(s) by inner fluctuations
- Majorana masses and SeeSaw mechanism for right-handed neutrinos (J. Barrett & A. Connes '06)

Constraints on the SM parameters at the cut-off Λ:

$$5\, g_1^2 = N_{SM}\, g_2^2 = N_{SM}\, g_3^2 = 3\, \frac{Y_2^2}{H}\, \frac{\lambda}{24} = \frac{3}{4}\, Y_2$$

- g_1, g_2, g_3: $U(1)_Y$, $SU_w(2)$, $SU_c(3)$ gauge couplings
- λ: quartic Higgs coupling
- Y_2: trace of the Yukawa matrix squared
- H: trace of the Yukawa matrix to the fourth power
- N_{SM}: number of standard model generations

Consequences from the SM constraints:

Input:

- Big Desert
- $g_1(m_Z) = 0.3575,\ g_2(m_Z) = 0.6514,\ g_3(m_Z) = 1.221$
- renormalisation group equations
- ($m_{top} = 171.2 \pm 2.1$ GeV)

Output:

- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17}$ GeV
- $m_{top} < 190$ GeV (Thumstädter, Tolksdorf 05)
- no 4^{th} SM generation

Excluded by Tevatron & LHC since:

- $m_{SMS} \neq 168.3 \pm 2.5$ GeV
- $\frac{5}{3} g_1(\Lambda)^2 \neq g_2(\Lambda)^2$

How unique is the Standard Model?

The aim: Classifying the internal spaces

$$\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$$

- with respect to the number of summands in the algebra
- with respect to physical criteria

Little Reminder

For the Standard Model we have

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$$

or alternatively

$$\mathcal{A}_f = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$$

Physicist's "shopping list" (B. Iochum, T. Schücker, C.S. '03):

The physical models emerging from the spectral action are required to

- be irreducible i.e. to have the smallest possible internal Hilbert space (minimal approach)
- allow a non-degenerate Fermionic mass spectrum
- be free of harmful anomalies
- have unbroken colour groups
- possess no uncharged massless Fermions

Classification Results
(B.Iochum, J.-H. Jureit, T.Schücker, C.S. 2003-2008):

# sum. in $\mathcal{A}_f$	KO 0	KO 6
1	no model	no model
2	no model	no model
3	SM[2]	no model
4	SM[2], el.-str.[1]	SM[2], el.-str.[1]
6		SM[2] + el.-str.[1], $2 \times$ el.-str.[1]

[1] Electro-Strong Model: "electron+proton", no Higgs,

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_n(\mathbb{C}),$$
$$G_{gauge} = U(1) \times SU(n)/SO(n)/Sp(n)$$

[2] first family, colour group = $SU(n)/SO(n)/Sp(n)$

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Beyond SM: the general strategy (bottom-up approach)

- find finite geometry that has SM as sub-model (tricky)
  => particle content, gauge group & representation
- make sure everything is anomaly free
- compute the spectral action => constraints on parameters
- determine the cut-off scale Λ with suitable sub-set of the constraints
- use renorm. group equations to obtain low energy values of (hopefully) interesting parameters
  (Higgs couplings, Yukawa couplings)
- check with experiment! (and here we usually fail)

SM + $U(1)_X$ scalar field + new fermions (C.S. '09 & '13):

- SM as a sub-model: comme il faut!
- Internal space: $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus_{i=1}^{6} \mathbb{C}_i$
- gauge group: $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X$
- new fermions in each SM-generation:

$$X_l^1 \oplus X_l^2 \oplus X_l^3 : (0,1,1,+1) \oplus (0,1,1,+1) \oplus (0,1,1,0)$$
$$X_r^1 \oplus X_r^2 \oplus X_r^3 : (0,1,1,+1) \oplus (0,1,1,0) \oplus (0,1,1,+1)$$
$$V_\ell^w,\ V_r^w : \ (0,\bar{2},1,0)$$
$$V_\ell^c,\ V_r^c : \ (-1/6,1,\bar{3},0)$$

- new scalar: $\sigma : (0,1,1,+1)$

The Lagrangian (scalar potential & new terms):

- $\mathcal{L}_{scalar} = -\mu_1^2|H|^2 - \mu_2^2|\sigma|^2 + \frac{\lambda_1}{6}|H|^4 + \frac{\lambda_2}{6}|\sigma|^4 + \frac{\lambda_3}{3}|H|^2|\sigma|^2$

- $\mathcal{L}_{ferm} = g_{\nu,X^1}\,\bar{\nu}_r\sigma X^1_\ell + \bar{X}^1_\ell m_X X^1_r + g_{X^2}\,\bar{X}^2_\ell\sigma X^2_r$
  $+ g_{X^3}\,\bar{X}^3_\ell\sigma X^3_r + \bar{V}^c_\ell m_c V^c_r + \bar{V}^w_\ell m_w V^w_r + \ h.c.$

- $\mathcal{L}_{gauge} = \frac{1}{g_4^2} F_X^{\mu\nu} F_{X,\mu\nu}$

- Symmetry breaking:
  $U(1)_Y \times SU(2)_w \times SU(3)_c \times U(1)_X \to U(1)_{el.} \times SU(3)_c \times \mathbb{Z}_2$

The constraints at Λ:

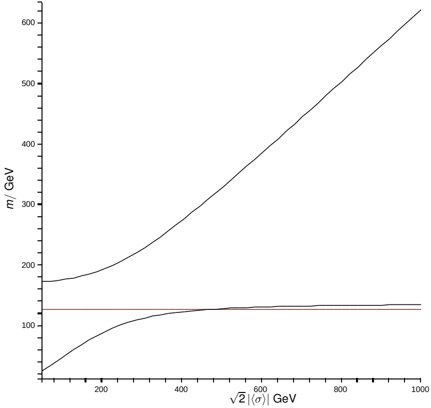
- $g_2(\Lambda) = g_3(\Lambda) = \sqrt{\frac{7}{6}}\, g_1(\Lambda) = \sqrt{\frac{4}{3}}\, g_4(\Lambda)$
- $\lambda_1(\Lambda) = 36\, \frac{H}{Y_2}\, g_2(\Lambda)^2, \ \ \lambda_2(\Lambda) = 36\, \frac{tr(g^4_{\nu,X^1})}{tr(g^2_{\nu,X^1})^2}\, g_2(\Lambda)^2$
- $\lambda_3(\Lambda) = 36\, \frac{tr(g^2_{\nu})}{Y_2}\, g_2(\Lambda)^2$
- $Y_2(\Lambda) = tr(g^2_{\nu,X^1})(\Lambda) + tr(g^2_{X^1})(\Lambda) + tr(g^2_{X^2})(\Lambda) = 6\, g_2(\Lambda)^2$

Some simplifications:

- $Y_2 \approx 3g_{top} + g_{\nu_\tau}$
- $tr(g^2_{X^1})(\Lambda) \approx tr(g^2_{X^2})(\Lambda) \approx 0$
- $tr(g^2_{\nu,X^1})(\Lambda) \approx g_{\nu,X}(\Lambda)^2 = 6\, g_2(\Lambda)^2$
- $(m_w)_{ij} \approx \Lambda$, $(m_c)_{ij} \approx 10^{15}$ GeV

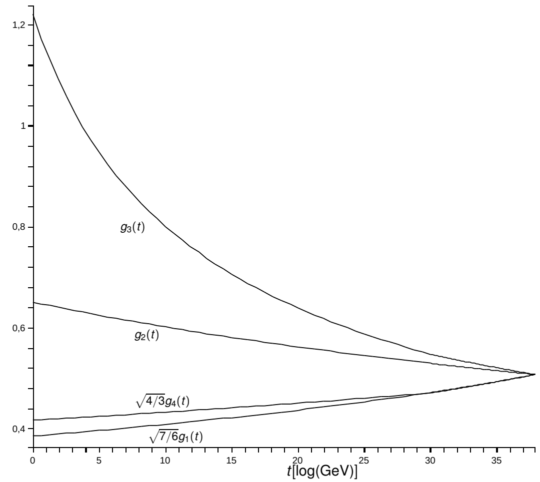
Results for 1-loop renormalisation groups:

- Constraints
  => $\Lambda \approx 2 \times 10^{18}$ GeV
- $m_{top} \approx 172.9 \pm 1.5$ GeV
- $m_{\sigma_{1,SMS}} \approx 125 \pm 1.1$ GeV
- $m_{\sigma_2} \approx 445 \pm 139$ GeV
- $m_{Z_X} \approx 254 \pm 87$ GeV
- $g_4(m_Z) \approx 0.36$
- $m_{X_2,X_3} \lesssim 50$ GeV
- free parameter: $|\langle\sigma\rangle|$



Mass EVs of scalar fields

Running of the gauge couplings with normalisation factors



Further promising alternatives:

- Grand symmetric models + Spectral Action
  Devastato, Lizzi, Martinetti
- Pati-Salam type models + Spectral Action
  Chamseddine, Connes, van Suijlekom
- Non-associative "Spectral Triples"
  Boyle, Farnsworth, Wulkenhaar
- Pauli-Dirac-Yukawa operators on Clifford module bundles + Wodzicki residue as bosonic action
  Ackermann, Thumstädter, Tolksdorf et al.

Note: Following Tolksdorf et al. the Chamseddine-Connes Dirac operator can be considered to be a generalised Dirac operator in the sense of Quillen / Berline, Getzler and Vergne.

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Questions & to-do-list

- Is the SM + scalar model compatible with LHC and BICEP/Planck data?
- Does the SM + scalar model contain viable dark matter candidates?
- Explore parameter space (g_{ν,X^1}, $g^2_{X^2}$, $g^2_{X^3}$, m_{X^1}, m_{V^w}, m_{V^c})
- Extend renormalisation group analysis to n-loop, $n \geq 2$
- Is the geometry a "sub-geometry" of a Connes-Chamseddine-type geometry?
- Classify Models beyond the Standard Model