The role of BRST charge as a generator of gauge transformations in quantization of gauge theories and Gravity

T. P. Shestakova

Department of Theoretical and Computational Physics,

Southern Federal University,

Sorge St. 5, Rostov-on-Don 344090, Russia

E-mail: shestakova@sfedu.ru

In the Batalin – Fradkin – Vilkovisky (BFV) approach to quantization of gauge theories a principal role is given to the BRST charge since BRST invariant quantum states are believed to be physical states.

In the BFV approach the BRST charge can be constructed as a series in Grassmannian (ghost) variables with coefficients given by generalized structure functions of constraints algebra:

$$\Omega_{BFV} = \int d^3x \left(c^{\alpha} U^{(0)}_{\alpha} + c^{\beta} c^{\gamma} U^{(1)\alpha}_{\gamma\beta} \rho_{\alpha} + \ldots \right)$$

c^a, ρ_a are the BFV ghosts and their conjugate momenta, $U^{(n)}$ are nth order structure functions, while zero order structure functions $U_{\alpha}^{(0)}=G_{\alpha}$ are Dirac secondary constraints.

In quantum theory physical states are annihilated by the BRST charge $\hat{\Omega}\,|\Psi\rangle = 0$

which is believed to be equivalent to the quantum version of constraints $\hat{G}_{\alpha}\,|\Psi\rangle = 0$

M. Hennaux, *Phys. Rep.* **126** (1985), P 1-66.

There exist another way to construct the BRST charge making use of global BRST symmetry and the Noether theorem.

The Faddeev − Popov action for the Yang − Mills fields in the Lorentz gauge is known to be BRST invariant:

$$S_{YM}=\int d^4x\left[-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}-i\bar{\theta}_a\partial^\mu D_\mu\theta^a+\pi_a\partial^\mu A^a_\mu\right]$$

$$\Omega_{Noether}=\int d^3x\left[\frac{\partial L}{\partial(\partial_0\varphi^a)}\delta\varphi^a+\frac{\partial L}{\partial(\partial_0\partial_\mu\varphi^a)}\delta(\partial_\mu\varphi^a)-\partial_\mu\left(\frac{\partial L}{\partial(\partial_0\partial_\mu\varphi^a)}\right)\delta\varphi^a\right]$$

$$\Omega_{YM}=\int d^3x\left(-\theta^a D_i p^i_a-i\pi_a P^a+\frac{1}{2}\bar{P}_a g f^a_{bc}\theta^b\theta^c\right)$$

The BRST charge for the Yang − Mills fields constructing according to the Noether theorem coincides exactly with the one obtained by the BFV prescription after replacing the BFV ghosts by the Faddeev − Popov ghosts.

In the case of gravity we deal with space-time symmetry, and we should take into account explicit dependence of the Lagrangian and the measure on space-time coordinates.

$$\Omega_{grav}=\int d^3x\left[\frac{\partial L}{\partial(\partial_0\varphi^a)}\delta\varphi^a+\frac{\partial L}{\partial(\partial_0\partial_\mu\varphi^a)}\delta(\partial_\mu\varphi^a)-\partial_\mu\left(\frac{\partial L}{\partial(\partial_0\partial_\mu\varphi^a)}\right)\delta\varphi^a+\partial_0\left(Lx^0\right)\right]$$

The isotropic model:

$$S_{isotr}=\int dt\left[-\frac{1}{2}\frac{a\dot{a}^2}{N}+\frac{1}{2}Na+\lambda\left(\dot{N}-\frac{df}{da}\dot{a}\right)+\bar{\theta}\frac{d}{dt}\left(-\dot{N}\theta-N\dot{\theta}+\frac{df}{da}\dot{a}\theta\right)\right]$$

One can check that the action for this model is not invariant under BRST transformations. However, the BRST invariance can be restored by adding to the action the additional term

$$S_1=\int dt\frac{d}{dt}\left[\bar{\theta}\left(\dot{N}-\frac{df}{da}\dot{a}\right)\theta\right]$$

T. P. Shestakova, *Class. Quantum Grav.* **28** (2011), 055009.

The isotropic model:

$$S_{isotr}=\int dt\left[-\frac{1}{2}\frac{a\dot{a}^2}{N}+\frac{1}{2}Na+\lambda\left(\dot{N}-\frac{df}{da}\dot{a}\right)+\bar{\theta}\frac{d}{dt}\left(-\dot{N}\theta-N\dot{\theta}+\frac{df}{da}\dot{a}\theta\right)\right]$$

The BRST charge constructed according to the Noether theorem: $\Omega_{isotr}=-H\theta-\pi P$

H is a Hamiltonian in extended phase space,

$$H=-\frac{1}{2}\frac{N}{a}\left[p^2+2p\pi\frac{df}{da}+\pi^2\left(\frac{df}{da}\right)^2\right]-\frac{1}{2}Na+\frac{1}{N}\bar{P}P \qquad \pi=\lambda+\dot{\bar{\theta}}\theta$$

The important features of the proposed approach to Hamiltonian dynamics in extended phase space:

- Thanks to the differential form of gauge condition, the Hamiltonian can be obtained by usual rule $H=p\dot{a}+\bar{P}\dot{\theta}+\dot{\bar{\theta}}P-L$
- Hamiltonian equations in extended phase space are fully equivalent to Lagrangian equations, constraints and gauge conditions being true Hamiltonian equations.

V. A. Savchenko, T. P. Shestakova and G. M. Vereshkov, *Grav. Cosmol*. **7** (2001), P. 18-28.

V. A. Savchenko, T. P. Shestakova and G. M. Vereshkov, *Grav. Cosmol*. **7** (2001), P. 102-116.

The BRST charge constructed according to the Noether theorem

$$\Omega_{isotr} = -H\theta - \pi P$$

generates correct transformations for all degrees of freedom, including gauge ones:

$$\delta N = \{N, \Omega_{isotr}\} = -\frac{\partial H}{\partial \pi}\theta - P = -\dot{N}\theta - N\dot{\theta} \qquad \dot{N} = \frac{\partial H}{\partial \pi}$$

The BRST charge constructed according to the BFV prescription

$$\Omega_{isotr}^{BFV} = -T\theta - \pi P \qquad T = -\frac{1}{2a}p^2 - \frac{1}{2}Na$$

$$\hat{\Omega}\,|\,\Psi\rangle = 0 \quad \Rightarrow \quad \hat{T}\,|\,\Psi\rangle = 0$$ (the Wheeler − DeWitt equation).

The BFV charge fails to produce a correct transformation for the gauge variable *N*. At the same time, for the Noether charge the condition of BRST invariance of quantum states together with the requirement of hermicity of Hamiltonian operator does not lead to the Wheeler − DeWitt equation.

We face the contradiction:

On the one hand, at the classical level we have a mathematically consistent formulation of Hamiltonian dynamics in extended phase space which is equivalent to the Lagrangian formulation of the original theory, and the BRST generator constructed in accordance with the Noether theorem, that produces correct transformation for all degrees of freedom.

On the other hand, at the quantum level our approach appears to be not equivalent to the BFV approach as well as the Dirac quantization scheme.

In the situation when an experiment cannot indicate what way is correct, what should one prefer?

The generalized spherically-symmetric gravitational model with the metric:

$$ds^2 = \left[-N^2(t,r) + (N^r(t,r))^2 V^2(t,r)\right]dt^2 + 2N^r(t,r)V^2(t,r)dt\,dr$$
$$+V^2(t,r)dr^2 + W^2(t,r)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The sum of gauge-fixing and ghost parts of the action is not invariant under BRST transformations:

$$S_{(gauge)} = \int dt \int_0^\infty dr \left[\lambda_0\left(\dot{N} - \frac{\partial f}{\partial V}\dot{V} - \frac{\partial f}{\partial W}\dot{W}\right) + \lambda_r\left(\dot{N}^r - \frac{\partial f^r}{\partial V}\dot{V} - \frac{\partial f^r}{\partial W}\dot{W}\right)\right]$$

$$S_{(ghost)} = \int dt \int_0^\infty dr \left[\bar{\theta}_0 \frac{d}{dt}\left(-\dot{N}\theta^0 - N'\theta^r - N\dot{\theta}^0 + NN^r(\theta^0)'\right.\right.$$
$$\left. - \frac{\partial f}{\partial V}\left[-\dot{V}\theta^0 - V'\theta^r - V(\theta^r)' - VN^r(\theta^0)'\right] - \frac{\partial f}{\partial W}\left[-\dot{W}\theta^0 - W'\theta^r\right]\right)$$
$$+\bar{\theta}_r \frac{d}{dt}\left(-\dot{N}^r\theta^0 - (N^r)'\theta^r - N^r\dot{\theta}^0 - \dot{\theta}^r + N^r(\theta^r)' + \frac{N^2}{V^2}(\theta^0)' + (N^r)^2(\theta^0)'\right.$$
$$\left.\left. - \frac{\partial f^r}{\partial V}\left[-\dot{V}\theta^0 - V'\theta^r - V(\theta^r)' - VN^r(\theta^0)'\right] - \frac{\partial f^r}{\partial W}\left[-\dot{W}\theta^0 - W'\theta^r\right]\right)\right]$$

The generalized spherically-symmetric gravitational model:

To ensure BRST invariance of the action we have to add the following terms

$$S_2=\int dt\int_0^\infty dr\left(\frac{d}{dt}\left[\bar\theta_0\left(\dot N-\frac{\partial f}{\partial V}\dot V-\frac{\partial f}{\partial W}\dot W\right)\theta^0\right]+\frac{d}{dr}\left[\bar\theta_0\left(\dot N-\frac{\partial f}{\partial V}\dot V-\frac{\partial f}{\partial W}\dot W\right)\theta^r\right]\right.$$
$$\left.+\frac{d}{dt}\left[\bar\theta_r\left(\dot N^r-\frac{\partial f^r}{\partial V}\dot V-\frac{\partial f^r}{\partial W}\dot W\right)\theta^0\right]+\frac{d}{dr}\left[\bar\theta_r\left(\dot N^r-\frac{\partial f^r}{\partial V}\dot V-\frac{\partial f^r}{\partial W}\dot W\right)\theta^r\right]\right)$$

The BRST charge constructed according to the Noether theorem is

$$\Omega=\int dr\left[-H\theta^0-P_V V'\theta^r-P_N\frac{\partial f}{\partial V}V'\theta^r-P_{N^r}\frac{\partial f^r}{\partial V}V'\theta^r-P_W W'\theta^r-P_N\frac{\partial f}{\partial W}W'\theta^r-P_{N^r}\frac{\partial f^r}{\partial W}W'\theta^r\right.$$
$$-P_V VN^r(\theta^0)'-P_N\frac{\partial f}{\partial V}VN^r(\theta^0)'-P_{N^r}\frac{\partial f^r}{\partial V}VN^r(\theta^0)'-P_V V(\theta^r)'-$$
$$\left.-P_N\frac{\partial f}{\partial V}V(\theta^r)'-P_{N^r}\frac{\partial f^r}{\partial V}V(\theta^r)'-\bar P_{\theta^0}(\theta^0)'\theta^r-\bar P_{\theta^r}(\theta^r)'\theta^r-P_N P_{\bar\theta_0}-P_{N^r}P_{\bar\theta_r}-\frac{NWW'(\theta^0)'}{V}\right]$$

T. P. Shestakova, *Grav. Cosmol.* **20** (2014), P. 67-79.

The full gravitational theory:

One can use gauge conditions in a general form, $f^{\mu}(g_{\nu\lambda})=0$

The gauge fixing and ghost parts of the action will be

$$S_{(gauge)}=\int d^4x\,\lambda_{\mu}\frac{d}{dt}f^{\mu}(g_{\nu\lambda})=\int d^4x\,\lambda_{\mu}\left(\frac{\partial f^{\mu}}{\partial g_{00}}\dot{g}_{00}+2\frac{\partial f^{\mu}}{\partial g_{0i}}\dot{g}_{0i}+\frac{\partial f^{\mu}}{\partial g_{ij}}\dot{g}_{ij}\right)$$

$$S_{(ghost)}=-\int d^4x\,\bar{\theta}_{\mu}\frac{d}{dt}\left[\frac{\partial f^{\mu}}{\partial g_{\nu\lambda}}\left(\partial_{\rho}g_{\nu\lambda}\theta^{\rho}+g_{\lambda\rho}\partial_{\nu}\theta^{\rho}+g_{\nu\rho}\partial_{\lambda}\theta^{\rho}\right)\right]$$

The additional term ensuring BRST invariance of the action in this general case reads

$$S_3=\int d^4x\,\partial_{\mu}\left[\bar{\theta}_{\nu}\frac{d}{dt}f^{\nu}(g_{\lambda\rho})\theta^{\mu}\right]$$

One can inquire about a physical meaning of the selection rules.

In quantum field theory with asymptotic states their meaning is clear: in asymptotic states interactions are negligible, and these states must not depend on gauge and ghost variables which are considered as non-physical. But ghost fields cannot be excluded in an interaction region.

In the gravitational theory, except some few situations, we need to explore states inside the interaction region, like inside a closed universe with non-zero curvature, let alone more complicated topology.

Then, what would be a definition of physical states in such cases?

Today we have no satisfactory answer for this question.