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Quantum field theory on a quantum space-time: Hawking radiation and the Casimir effect

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Plan

- Review of quantum theory of spherically symmetric vacuum space-times.
- The matter Hamiltonian as a parameterized Dirac observable.
- Quantum vacua.
- Hawking radiation.
- Casimir effect.

Summary:

We have recently found, in closed form, the space of physical states corresponding to spherically symmetric vacuum space-times in loop quantum gravity.

We wish to consider the quantization of a test scalar fields on such quantum space-times.

The idea will be to represent the matter part of the Hamiltonian constraint as a parameterized Dirac observable for the gravitational variables and we can therefore evaluate its expectation value on states of the physical space of states of vacuum gravity.

We choose states very peaked around a Schwarzschild space-time of a given mass. The resulting expectation value of the matter part of the Hamiltonian constraint becomes a classical Hamiltonian, quantum corrected due to the quantum background space time. We proceed to quantize such Hamiltonian in the traditional way, defining modes and creation and annihilation operators and obtain its vacua. We then compute the Hawking radiation.

Main result: the quantum background space-time acts as a lattice discretization of the field theory, naturally regulating it and eliminating infinities, but otherwise changing in small but important ways the traditional picture of QFT on CST.

The quantum background: vacuum spherically symmetric LQG

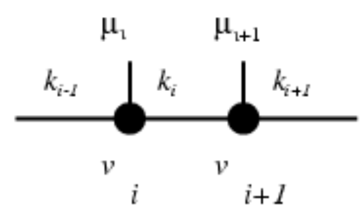
After a rescaling and combination of the constraints that turns their algebra into a Lie algebra, we were able to solve in closed form for the space of physical states of spherically symmetric vacuum LQG (RG, JP PRL 110, 211301)

We use the variables adapted to spherical symmetry developed by Bojowald and Swiderski (CQG23, 2129 (2006)). One ends up with two canonical pairs, E^x, E^φ, K_x, K_φ.

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\mathrm{sign}(E^x)\frac{(E^\varphi)^2}{\sqrt{|E^x|}} K_x \qquad K_{\theta\theta} = -\sqrt{|E^x|}\frac{A_\varphi}{2\gamma},$$

Kinematical states are given by one dimensional spin networks,

$$T_{g,\vec{k},\vec{\mu}}(K_x, K_\varphi) = \langle K_x, K_\varphi \Big|$$  $$\Big\rangle$$

$$= \prod_{e_j \in g} \exp\left(\frac{i}{2} k_j \int_{e_j} K_x(x) dx\right) \prod_{v_j \in g} \exp\left(\frac{i}{2}\mu_j \gamma K_\varphi(v_j)\right)$$

A basis of he physical states are given by $|\tilde{g},\vec{k},M>$ with $\tilde{g}$ a diffeo equivalent class of one dimensional graphs, the k's are proportional to the eigenvalues of the areas of symmetry and M is the ADM mass.

More details in: RG, J. Olmedo, JP, CQG 31 095009 (2014) arXiv:1310.5996

This constitutes the physical space of states for pure gravity. We now want to study a quantum field living on this quantum state. For the combined system we assume the states have the form of a direct product between the gravity and the matter states.

We will represent the matter part of the Hamiltonian constraint as a parameterized Dirac observable of the gravitational degrees of freedom. This will allow to promote it to an operator that is well defined on the physical space of states.

Since some people may not be familiar with this technique we will review it a bit.

Parameterized Dirac observables:

People are familiar with Carlo's evolving constants of the motion. They are Dirac observables that are function of a parameter.

$$\{Q_i(t), \phi_\alpha\} \approx 0 \qquad Q_i(t, q^a, p_a)\big|_{t=q1} = q_i$$

For instance, for the relativistic particle.
Two independent observables:

$$\phi = p_0^2 - p^2 - m^2$$

$$p, X \equiv q - \frac{p}{\sqrt{p^2 + m^2}} q^0, \qquad Q(t, q^a, p_a) = X + \frac{p}{\sqrt{p^2 + m^2}} t$$

$$Q(t = q^0, q^a, p_a) = q$$

What is perhaps less well known is that they are a mechanism for representing gauge dependent quantities via Dirac observables. Their value is not well defined: it depends on the value of the parameter. Choosing a parameter is tantamount to choosing a gauge, and therefore the quantity becomes gauge invariant.

For instance $E^x(x)$ can be promoted to a parameterized observable,

One defines a Dirac observable O(z) z in [0,1]

$$\hat{O}(z)|\vec{k},\tilde{g}\rangle_{\rm phys} = \ell^2_{\rm Planck} k_{{\rm Int}(V z)}|\vec{k},\tilde{g}\rangle_{\rm phys},$$

$$\hat{E}^x(x)|\vec{k},\tilde{g}\rangle_{\rm phys} = \hat{O}(z(x))|\vec{k},\tilde{g}\rangle_{\rm phys}.$$

Where z(x) is the functional parameter that embodies the fact that $E^x(x)$ is gauge dependent.

One can also write a parameterized Dirac observable for the metric (e.g. tx component)

$$g_{tx} = g_{xx}N_r = -\frac{(E^x)'\,K_\varphi}{2\sqrt{E^x}\sqrt{1+K_\varphi^2-\frac{2GM}{\sqrt{E^x}}}},$$

Here the functional parameter is K_φ, which corresponds to a choice of slicing. For instance, for usual Schwarzschild coordinates $K_\varphi=0$, and a non-vanishing K_φ can be used to consider horizon penetrating coordinates, like Eddington-Finkelstein

The main effect of considering the quantum vacuum is that the equations for the scalar field become similar to those of a scalar field discretized on a lattice. The lattice in this case is provided by the (one dimensional) spin network state of the background space-time.

$$\hat{H}_i = \hat{A}_i P_{\phi,i}^2 + \hat{B}_i \left(\phi_{i+1} - \phi_i\right)^2 + \hat{C}_i P_{\phi,i} \left(\phi_{i+1} - \phi_i\right)$$

For states with equally spaced nodes and $z(x)=x/x_{max}$,

$$\hat{A}_i|\psi,\tilde{g},\vec{k}\rangle_{\rm grav} = \frac{1}{2}\frac{\left[\sqrt{|\Delta k_i+1|}-\sqrt{|\Delta k_i-1|}\right]^2}{\ell^2_{\rm Planck}}\frac{\left(1+\mathcal{K}^2_{\varphi,i}-\frac{2GM}{\sqrt{\ell^2_{\rm Planck}k_i}}\right)}{\sqrt{\ell^2_{\rm Planck}k_i}}|\psi,\tilde{g},\vec{k}\rangle_{\rm grav},$$

$$\hat{B}_i|\psi,\tilde{g},\vec{k}\rangle_{\rm grav} = 2\frac{\left[\sqrt{|\Delta k_i+1|}-\sqrt{|\Delta k_i-1|}\right]^2}{\ell^2_{\rm Planck}}\left(1+\mathcal{K}^2_{\varphi,i}-\frac{2GM}{\sqrt{\ell^2_{\rm Planck}k_i}}\right)k_i^{3/2}\ell^3_{\rm Planck}|\psi,\tilde{g},\vec{k}\rangle_{\rm grav},$$

$$\hat{C}_i|\psi,\tilde{g},\vec{k}\rangle_{\rm grav} = 2\frac{\left[\sqrt{|\Delta k_i+1|}-\sqrt{|\Delta k_i-1|}\right]^2}{\ell^2_{\rm Planck}}\sqrt{1+\mathcal{K}^2_{\varphi,i}-\frac{2GM}{\sqrt{\ell^2_{\rm Planck}k_i}}}\mathcal{K}_{\varphi,i}\sqrt{\ell^2_{\rm Planck}k_i}|\psi,\tilde{g},\vec{k}\rangle_{\rm grav}.$$

The spacing in the lattice is given by the condition of the quantization of the area of the surfaces of spherical symmetry. That condition implies that the points are separated a distance at least $L^2_{Planck}/(4GM)$ in the exterior of the black hole.

As a consequence, the discrete equations become excellent approximations of the continuum equations at energies lower than the Planck energy, and most calculations follow like those in the continuum.

One can proceed to define modes and in terms of them creation and annihilation operators and compute the vacua. The calculations of the Unruh, Boulware and Hartle-Hawking vacua resemble those of the continuum with very small corrections.

The main change is that certain trans Planckian modes that would have wavelengths smaller than the lattice spacing are suppressed. This implies that physical quantities that may diverge at horizons, like the stress energy tensor, remain finite. This may have implications for future attempts to do back reaction calculations.

The canonical equations for the scalar field correspond to a spatially discretized version of the Klein-Gordon equation in curved space time.

$$(\sqrt{-g}g^{ab}\phi_{,a})_{,b} = 0$$

The construction of quantum vacua is carried out considering modes that solve the wave equation and creation and anhilation operators for these modes. We will only sketch the properties fo the Boulware modes

$$z(x_j^*) = \frac{x(x_j^*)}{N_V \Delta} \qquad x_j^* = x_j + 2GM \ln(\frac{x_j}{2GM} - 1)$$

$$\partial_0^{\,2}\phi(x_j^*,t) - [\frac{\phi(x_{j+1}^*,t) - \phi(x_j^*,t)}{\Delta_j^{\,2}} - \frac{\phi(x_j^*,t) - \phi(x_{j-1}^*,t)}{\Delta_j \Delta_{j-1}}] = 0$$

$$\Delta_j = \Delta + \frac{2GM}{j}$$

Asymptotically $\Delta_j \to \Delta$ and one recovers an excellent aproximation to the Boulware vacuum.

Asymptotically, near scri$_-$ and scri$_+$ the modes are:

$$f_\omega = \frac{1}{\sqrt{2\pi\omega}} \exp(-i\omega_n t - ik_n x^*) \qquad g_\omega = \frac{1}{\sqrt{2\pi\omega}} \exp(-i\omega_n t + ik_n x^*)$$

with

$$\omega_n = \sqrt{\frac{2-2\cos(k_n\Delta)}{\Delta^2}} \text{ and } k_n = \frac{2\pi n}{(N_V - i_H)\Delta}$$

Near to the horizon trans Planckian modes are heavily suppressed due to the discreteness of the spin network state. In our treatment there are no arbitrary frequency trans-Planckian modes, the dispersion relation is modified in a sub-luminal way (it does not affect the horizon structure) and there are no singularities from physical quantities, like the expectation value of the stress energy tensor.

One can perform similar analyses for the Unruh and Hartle-Hawking vacua.

A point to be careful about is that our spherical treatment uses from the outset variables adapted to the symmetry that are time independent. The spin network vertices we have chosen are fixed in those slicings. When one considers the slicings required for Hartle-Hawking and Unruh one ends up with spacings that are time dependent in the slicings considered and the resulting discretized wave equations reflect this. This does not cause problems and we have shown that one reproduces the usual results (more details in the paper).

Calculation of Hawking radiation

The calculation of the Hawking radiation proceeds in the usual way, through the computation of the Bogoliubov coefficients.

$$\beta_{i_1,k} = \left(u_{i_1}^{\mathrm{out}}, u_k^{\mathrm{in}\,*}\right)$$

With the states

$$\begin{aligned} u^{\mathrm{in}}_{\omega,\ell,m} &= \frac{1}{\sqrt{4\pi\omega}} \frac{\exp\left(-i\omega U\right)}{r} Y_\ell^m\left(\theta,\varphi\right) \\ u^{\mathrm{out}}_{\omega,\ell,m} &= \frac{1}{\sqrt{4\pi\omega}} \frac{\exp\left(-i\omega u\right)}{r} Y_\ell^m\left(\theta,\varphi\right) \end{aligned}$$

and u=t-x*, and U=-exp(u/(4GM)).

The usual calculation (modified to computing it with the spherical modes) yields an expression for the number operator of the out photons in terms of the in states of the form,

$$\langle \mathrm{in}|N^{\mathrm{out}}_{i_1,i_2}|\mathrm{in}\rangle = \frac{t_{\ell_1}(\omega_1)t^*_{\ell_2}(\omega_2)\delta(\omega_1-\omega_2)}{2\pi\sqrt{\omega_1\omega_2}}\int_{-\infty}^{\infty} dz \exp\left(-i\frac{(\omega_1-\omega_2)}{2}z\right)\left(\frac{\kappa}{2}\right)^2 \frac{\delta_{\ell_1,\ell_2}\delta_{m_1,m_2}}{\sinh^2\left(\frac{\kappa}{2}\left(z-i\epsilon\right)\right)},$$

With $z=u_2-u_1$ and i_1,i_2 the labels associated with the in and out states.

This expression has problems at z=0, hence the addition of the $i\varepsilon$ term.
In our approach, discreteness leads to $|z|> L_{\mathrm{Planck}}$, so that problem is eliminated. This is similar to the heuristic cutoff that had been proposed by Agullo, Navarro-Salas, Olmo and Parker PRD80, 047503 (2009). The corrected expression for the Hawking radiation is,

$$\langle \mathrm{in}|N^{\mathrm{out}}_{i_1,i_2}|\mathrm{in}\rangle = \frac{|t_\ell(\omega)|^2}{\exp\left(2\pi\omega/\kappa\right)-1} - \frac{\kappa^2\ell_{\mathrm{Planck}}}{96\pi^3\omega}$$

Notice that it is remarkable that the cutoff that arises naturally is Lorentz invariant.

The Casimir effect on a quantum geometry

To compute the Casimir force we will need to compute the integral of the expectation value of the T00 component of the stress energy in the region between the two shells, integrate it, and compute its derivative with respect to the separation of the shells. We will assume the shells are very far away from the origin (or black hole) to be able to ignore the centrifugal potential.

We consider a conformally coupled massless scalar field. The relevant component of the (improved) energy momentum tensor is,

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{6}\left(\phi'\right)^2 - \frac{1}{3}\phi\ddot{\phi}.$$

We begin by considering the spherical modes for a scalar field,

$$u_{n,l,m} = \exp(-i\omega(t+r))Y_{\ell,m}(\theta,\varphi)/(\sqrt{4\pi\omega}r),$$

And imposing Dirichlet boundary conditions at the shells, the fields take the form,

$$\phi\left(t,r_j\right) = \sqrt{\frac{2\pi}{N_I\Delta}}\sum_{n=0}^{\frac{N_I}{2}}\sum_{\ell=0}^{\frac{\pi n r_0}{L}}\sum_{m=-\ell}^{\ell}\left[\frac{a_{n,\ell,m}e^{-i\omega_n t}}{\sqrt{4\pi\omega_n}}\frac{\sin\left(\frac{2\pi n j\Delta}{N_I\Delta}\right)}{r_j}Y_{\ell,m}\left(\theta,\varphi\right) + \frac{a^{\dagger}_{n,\ell,m}e^{i\omega_n t}}{\sqrt{4\pi\omega_n}}\frac{\sin\left(\frac{2\pi n j\Delta}{N_I\Delta}\right)}{r_j}Y^{*}_{\ell,m}\left(\theta,\varphi\right)\right]$$

Which corresponds to a spherical sector from r_0 to r_0+L with $N_I\,\Delta=L$

The dispersion relation is typical of a lattice $\omega_n^2 = (2 - 2\cos(k_n\Delta))/\Delta^2$ and $k_n = (2\pi n)/(N_I\Delta)$.
and the creation and annihilation operators have the usual commutation relations.

To compute the expectation value of the stress tensor we need to compute radial and time derivatives of the field. We start from Green's function,

$$G_+^L(x, x') = \langle 0_L|\phi(x)\phi(x')|0_L\rangle.$$

And replacing the fields of the previous slide we get, after taking θ and φ coincident,

$$G_+^L(r, t; r', t') \sim \frac{L}{\pi}\int_{\pi/L}^{\pi/\Delta} dk \frac{1}{2\omega_k \Delta N_I} \frac{e^{-i\omega_k(t-t')}}{rr'} \left[\cos\left(k\left(z - z'\right)\right) - \cos\left(k\left(z + z'\right)\right)\right],$$

Where the sums have been approximated by integrals and the proportionality is given by a finite angular factor.

From there we then compute,

$$
\begin{aligned}
\langle 0_L|\dot{\phi}^2|0_L\rangle &= \frac{\partial^2}{\partial t\,\partial t'}G_+^L\bigg|_{(r,t)=(r',t')} = \frac{1}{2\pi}\frac{\pi r^2}{4L^2}\int_{\pi/L}^{\pi/\Delta} dk\frac{\omega_k}{r^2}\left[1+\cos\left(2kz\right)\right] \\
&= \frac{\pi r^2}{4L^2}\left[\frac{2}{\pi r^2\Delta^2}-\frac{\pi}{4r^2L^2}-\frac{1}{4r^2L^2}\left(\frac{L^2\sin^2\left(\pi z/L\right)}{\pi z^2}-\frac{L\sin\left(2\pi z/L\right)}{z}\right)+O\left(\Delta^2\right)\right], \\
\langle 0_L|\left(\phi'\right)^2|0_L\rangle &= \frac{\partial^2}{\partial r\,\partial r'}G_+^L\bigg|_{(r,t)=(r',t')} = \frac{1}{2\pi}\frac{\pi r^2}{4L^2}\int_{\pi/L}^{\pi/\Delta}\frac{dk}{\omega_k}\left(\frac{1}{r^4}\left[1-\cos\left(2kz\right)\right]+\frac{1}{r^2}\left[k^2+k^2\cos\left(2kz\right)\right]\right) \\
&= \frac{\pi r^2}{4L^2}\left[\frac{const.}{r^4}-\frac{\pi}{4r^2L^2}-\frac{32\mathrm{Li}_3\left(i\right)-i\pi^3-25\zeta\left(3\right)+16\pi G_c}{4\Delta^2\pi r^2L^2}\right. \\
&\quad\left.+\frac{1}{4r^2L^2}\left(\frac{L^2\sin^2\left(\pi z/L\right)}{\pi z^2}-\frac{L\sin\left(2\pi z/L\right)}{z}\right)+O\left(\Delta^2\right)\right] \\
\langle 0_L|\phi\ddot{\phi}|0_L\rangle &= \frac{\partial^2}{\partial\left(t'\right)^2}G_+^L\bigg|_{(r,t)=(r',t')} = -\langle 0_L|\dot{\phi}^2|0_L\rangle,
\end{aligned}
$$

And the stress energy tensor

$$
\begin{aligned}
\langle 0_L|T_{00}|0_L\rangle &= \frac{5}{6}\langle 0_L|\dot{\phi}^2|0_L\rangle+\frac{1}{6}\langle 0_L|\left(\phi'\right)^2|0_L\rangle = -\frac{\pi^2}{16L^4}-\frac{\pi}{24L^4}\left(\frac{L^2\sin^2\left(\pi z/L\right)}{\pi z^2}+\frac{L\sin\left(2\pi z/L\right)}{z}\right) \\
&\quad+O\left(\frac{1}{r^2}\right)
\end{aligned}
$$

With higher order terms all finite in the limit Δ->0

To get the effect we need to subtract the contribution one would get if no boundaries were present. For that effect we repeat the calculation for a spherical sector of large width L_M>L and subtract,

$$\langle 0_L|T_{00}|0_L\rangle_L - \langle 0_{L_M}|T_{00}|0_{L_M}\rangle_{L_M} = -\frac{\pi^2}{16L^4} - \frac{\pi}{24L^4}\left(\frac{L^2\sin^2(\pi z/L)}{\pi z^2} + \frac{L\sin(2\pi z/L)}{z}\right) + O\left(\frac{1}{L_M}\right).$$

We now integrate this from 0 to L in z to obtain the vacuum energy per unit area in the slab,

$$\text{Energy} = -\frac{\pi^2}{16L^3} - \frac{\text{Si}(2\pi)}{24} - \frac{-2\ln(2) - \gamma - \ln(\pi) + \text{Ci}(4\pi)}{48L^3} \sim \frac{0.03}{L^3},$$

This has the correct dependence on L and the numerical coefficient is about half the correct value. We attribute this to the crudeness of our approximation, in particular that we are truncating in ell. We are investigating better approximations right now.

Future work:

It is unrealistic to consider a background state highly peaked around the Schwarzschild mass, in general there will be some spread. That leads to a fuzziness in the lattice picture and the emergence of non-locality in the theory.

An important issue is that the presence of the discrete structure violates Lorentz invariance. This may improve by considering superpositions of backgrounds of different masses, but it may require restrictions on the type of matter fields that can be considered. Details coming soon.

The finiteness introduced by the lattice opens hopes for back reaction calculations. Work is under way, but it is not straightforward, since one needs to operate in the lattice and there are no easy approximations to work with.

More details at: arXiv:1312:3595

Summary:

- We can formulate quantum field theory in quantum space-times for fields on spherically symmetric gravity backgrounds.
- It approximates quantum field theory in curved space time very well.
- Discreteness naturally regularizes physical quantities, opening the possibility of back reaction calculations.
- Hawking radiation can be computed and the result coincides with previous heuristic results.
- The Casimir effect can be computed and the right dependence on separation is obtained.