Influence of quantum matter fluctuations on the expansion parameter of timelike geodesics

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Motivations

- At short distance the spacetime should be **non-commutative**.
- This feature should be encoded in the “**Quantum Gravity**”

  **No satisfactory description.**

- We can get information about such a theory analyzing **particular regimes** [Hawking].
- Gravity classically Matter by quantum theory.

\[ G_{ab}(x) = 8\pi \langle T_{ab}(x) \rangle_\omega \]

- **Doplicher, Fredenhagen and Roberts 95** use this to obtain **uncertainty relations** for the coordinates on a **flat** quantum space.
- **Starobinski** use this to obtain one of the first cosmological models with inflation.
Motivations

- **Semiclassical equations**: Quantum fields as source for classical ones, like:
  \[ G_{ab}(x) = \langle T_{ab}(x) \rangle. \]

- Fluctuations of \( T_{ab}(x) \) **diverge**. Cannot be renormalized.

- Smearing is needed: \( T_{ab}(f), \langle T_{ab}(f)^n \rangle \) give the **probability dist**.

- However, smearing **brakes covariance**.
  
  Solution: quantize the full theory.

- Intermediate step: **Langevin equation** (like Brownian motion).
  (Passive influence of the right side on the left one).

  \[ G_{ab} = T_{ab} \]
Carlip, Mosna and Pitelli \textbf{PRL} (2011) “Vacuum Fluctuations and the Small Scale Structure of Spacetime”.

- Effective 2d dilatonic model for gravity.
- Analyze the probability of a geodesic collapse at small scales.
- Expansion parameter of null geodesics.

\[
\dot{\theta} + \frac{1}{2} \theta^2 = -T
\]

- Probability distribution for a smeared energy density in a 2d CFT. \textbf{[Fewster Ford Roman 2010]}
  - Mean value vanishes.
  - It is bounded from below.
  - There is a long positive tail.
  - Negative energies are more likely.
Motivations

The Raychaudhuri equation for timelike geodesics provides a simplified model:

\[ \dot{\theta} + \frac{1}{3} \theta^2 = \ldots - (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \xi^\mu \xi^\nu \]

- It can be seen as a one-dimensional non-linear field theory.
- **Test of the ideas** in a simplified setting.
- Might provide hints on the underlying quantum gravity.
Plan of the talk

- Restriction of matter fields on timelike curves.
- Perturbative analysis of Raychaudhuri equation.
- Probability of focusing and some final comments on the arising probability distribution.
- Towards bounds for uncertainty of quantum coordinates.

This talk is based on

Restriction of Matter fields on timelike curves

Matter fields - Restriction on timelike curves

- Massless minimally coupled scalar quantum field.

\[-\Box \varphi = 0\]

- The quantization is very well under control.

- The $\ast$-$\ast$-algebra generated by linear fields $\varphi(f)$, implementing:

\[\varphi^*(f) = \varphi(\overline{f})\, , \quad [\varphi(f), \varphi(h)] = i\Delta(f, h)\, , \quad \varphi(\Box f) = 0\, .\]

- Assign to every spacetime [Brunetti Fredenhagen Verch]

\[M \mapsto \mathcal{A}(M)\]

- Local non linear fields can be added to the algebra. [Hollands Wald]
Extended algebra of fields

Following [Brunetti Fredenhagen Duetsch], $\mathcal{A}(M)$ algebra of functionals over smooth field configurations.

After deforming $\mathcal{A}(M) \Delta \rightarrow -2iH$ it can be extended trivially.

$$\mathcal{F}(M) := \{ F : \mathcal{E}(M) \rightarrow \mathbb{C} | F \text{ inf. diff. with compact support}, \quad WF(F^{(n)}) \cap (\overline{V}_+ \cup \overline{V}_-) = \emptyset \},$$

where the product is

$$F \star_H G := \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, H^{\otimes n} G^{(n)} \rangle$$

$H$ is an Hadamard parametrix, enjoying the microlocal spectrum condition.
Let be $\gamma \subset M$ a smooth timelike curve.

Not every element of $\mathcal{F}(M)$ can be tested on field configurations restricted on $\gamma$:

$$\mathcal{F}(M) \ni F(\varphi) \rightarrow \int \varphi \delta(\gamma) fd\mu, \quad F(\delta(\gamma)\varphi) \text{ diverges.}$$

We can define fields intrinsically on $\gamma$

$$\mathcal{F}(\gamma) := \{ F : \mathcal{E}(\gamma) \rightarrow \mathbb{C} | \text{ F inf. diff. with compact support, } \text{ } \}$$

$$\text{WF}(F^{(n)}) \cap (\mathbb{R}^n_+ \cup \mathbb{R}^n_-) = \emptyset \},$$

$$F \star_h G := \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, \ h^{\otimes n} G^{(n)} \rangle$$

being $h$ a two-point function with $\text{WF}(h) \subset \mathbb{R}_+ \times \mathbb{R}_-$. 
Restriction of Matter fields on timelike curves

Connection with the spacetime theory

**Question**

Can we imbed $\mathcal{F}(\gamma)$ into $\mathcal{F}(M)$?

Yes because we can restrict

$$h = H \circ (\gamma \otimes \gamma) = H \cdot \delta(\gamma \otimes \gamma)$$

$WF(\delta(\gamma \otimes \gamma))$ contains only spatial directions.

**Theorem**

Let $\iota_\gamma : \mathcal{E}(M) \to \mathcal{E}(\gamma)$ defined by $\iota_\gamma \varphi := \varphi \circ \gamma$ realizing the restriction of field configurations on $\gamma$

*Its pullback* imbed $\mathcal{F}(\gamma) \subset \mathcal{F}(M)$: $\iota_\gamma^* \mathcal{F}(\gamma) \subseteq \mathcal{F}(M)$.

$$\iota_\gamma^* F \star_H \iota_\gamma^* G = \iota_\gamma^*(F \star_h G),$$

It does **not** work on light like curves.
Raychaudhuri equation

- Consider a congruence of timelike geodesic $\mathcal{C}$.

  The **expansion parameter** $\theta$ measures the **rate of change** of $\frac{4}{3} \pi r^3$ along $\mathcal{C}$

  - $\theta > 0$ expansion
  - $\theta = 0$ parallel motion
  - $\theta < 0$ contraction

- Its evolution is governed by the **Raychaudhuri** equation

  $$\dot{\theta} = -\frac{1}{3} \theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu,$$

  $\omega_{\mu\nu}$: angular velocity of the geodesics;

  $\sigma_{\mu\nu}$: deformation parameter;

  $\xi^\mu$: tangent vector of the geodesic.
Einstein equation can be used to evaluate $R_{\mu\nu}$.

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$$

In the case of an expanding flat FRW spacetime

$$ds^2 = -dt^2 + a^2(t)dx^2, \quad \theta(t) = 3H(t)$$

Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3} \theta^2 - \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \xi^\mu \xi^\nu,$$

is equivalent to **Friedmann equations** (up to an initial condition).
Question

Can we treat fluctuations of the expansion parameter as fields in the matter algebra?

- The equation for $\psi \left( \theta = 3 \dot{\psi}/\psi \right)$ defined up to a scale.

$$
\ddot{\psi} + \frac{1}{3} \left( \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega^{\mu\nu} \omega_{\mu\nu} + T_{\text{cl}} \right) \psi + \frac{1}{3} \varphi^2 \psi = 0,
$$

$$
:= V
$$

We are interested in the fluctuations of $\psi$ induced by the ones of $\varphi$.

- We shall use perturbation theory and test if $\psi$ vanishes
  
  1. The fluctuations of $\omega_{\mu\nu}, \sigma_{\mu\nu}$ are negligible;
  2. The influence of $\psi$ on $\varphi$ is negligible.

- It is a one dimensional problem. It is a field theory on a line.
Retarded propagator of the theory

A poor man interacting quantum field theory.

\[ \ddot{\psi} + V \psi + \frac{1}{3} \dot{\phi}^2 \psi = 0. \]

The solution is formally

\[ \psi = \psi_0 + R_V (\dot{\phi}^2 \psi), \]

\( R_V: \mathcal{D}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}) \) the retarded propagator of \( P_\gamma = -\frac{d^2}{dt^2} - V \) i.e.

\[ R_V P_\gamma(f) = P_\gamma R_V(f) = f, \quad \text{supp}(R_V f) \subseteq J^+(\text{supp}(f)). \]

The integral kernel of \( R_V \) has the form

\[ R_V(x, y) = S(x, y) \theta(x - y), \quad (R_V f)(x) = \int \mathcal{E}(\mathbb{R}^2) R_V(x, y) f(y) dy. \]

We look for a recursive solution.
Perturbative analysis: Yang-Feldman method

$\dot{\varphi}^2 \rightarrow \lambda \varphi^2$. Solution as a **formal power series** in $\lambda$ around a **free classical solution** $\psi_0$.

$$
\psi(f) = \psi_0(f) + \psi_1(f) + \psi_2(f) + \ldots
$$

**[Epstein, Glaser, Steinmann, Hollands, Wald, Brunetti, Duetsch, Fredenhagen]** Choose $\lambda \in C_0^\infty(\gamma)$

$$
\psi_n(f) = R_V(\lambda \varphi^2 \psi_{n-1})(f) \quad n = 1, 2, \ldots
$$

$$
\psi_n(f) = \int f_R(x_{n-1}) S(x_{n-1}, x_{n-2}) \ldots S(x_1, x_0) \lambda(x_{n-1}) \ldots \lambda(x_0) \cdot
\underbrace{\vartheta(x_{n-1} - x_{n-2}) \ldots \vartheta(x_1 - x_0) \varphi^2(x_{n-1}) \ast h \ldots \ast h \varphi^2(x_0)}_{:= r(x_{n-1}, \ldots, x_0)}
$$

- To solve it we need to consider ill defined $R_V(x, y) \cdot h(x, y)$.
- We want $r$ for every possible $V \implies$ we leave $S$ out of $r$.
- **Small problem**, $S$ is not symmetric $\implies$ modify slightly the standard construction.
Construction of $r(x_n, \ldots, x_0)$ in $\mathcal{F}(\gamma)$

The $r(x_n, \ldots, x_0)$ are distributions with values in $\mathcal{F}(\gamma)$

1 retardation 1: if $x_n > \ldots > x_0$ then

$$r(x_n, \ldots, x_0) = \dot{\varphi}^2(x_n) \ast_h \ldots \ast_h \dot{\varphi}^2(x_0);$$

2 retardation 2: if it does not hold that $x_n \geq \ldots \geq x_0$ then

$$r(x_n, \ldots, x_0) = 0;$$

3 factorization: if $x_n \geq \ldots \geq x_0$ and $x_{m+1} > x_m$, $m \in \{1, \ldots, n\}$, then

$$r(x_n, \ldots, x_0) = r(x_n, \ldots, x_{m+1}) \ast_h r(x_m, \ldots, x_0);$$

4 initial element: $r(x_0) = \dot{\varphi}^2(x_0)$. 

Solution

The construction of $r$ is an application of the recently developed pAQFT. [Epstain, Glaser, Steinmann, Hollands, Wald, Brunetti, Duetsch, Fredenhagen, Rejzner]

**Inductive** construction of $r$ [Epstain Glaser] uses the previous general properties.

- We have the **initial element**.
- Suppose that you have all $rs$ with $n - 1$ entries then
  1. Construct $r(x_n, \ldots, x_0)$ outside the full diagonal $x_n = \ldots = x_0$ with the **factorization property**.
  2. Extend it to the full diagonal by means of Steinmann scaling degree techniques [Brunetti Fredenhagen].

In the last step there is the usual renormalization freedom expressed by a certain number of constants.
Adiabatic limit

- With those \( r \) we can obtain \( \psi_n(f) \in \mathcal{F}(\gamma) \) for every \( n \).

- The last step is the analysis of the limit \( \lambda \to 1 \) (in \( \mathcal{F}(\gamma) \)).

- It can be performed in \( \mathcal{F}(\gamma) \) because the equation for \( \psi \) is linear in \( \psi \) and we smear \( \psi \) with a compactly supported smooth function \( f \).

- Formally we can split \( \psi = \psi^+ + \psi^- \)

\[
\ddot{\psi}^\pm + V\psi^\pm + \frac{1}{3}\dot{\phi}^2\psi^\pm = \pm b,
\]

- \( b \) smooth and supported in the past of \( f \).
- \( \text{supp}(\psi^\pm) \) in the future/past of \( \text{supp}(b) \).

For \( \psi^+ \) with \( \lambda = 1 \) the retarded integral are compact.
With those $r$ we can obtain $\psi_n(f)$ for every $n$ in the limit $\lambda = 1$.

**Question**

What kind of fields are $\psi_n(f)$?

**Theorem**

$\psi_n(f)$ are functionals over matter field configurations. They are elements of $\mathcal{F}(\gamma) \forall n$.

- The perturbative analysis of the moments of $\psi$ can be put on firm mathematical grounds.

- If we have a state $\omega$ for the matter fields, we can construct the probability distribution for $\psi(f)$. 
Application in Minkowski

- Estimate the focusing probability of a family of \textbf{timelike parallel geodesics} on Minkowski within the interval of time $I$.

  (collapse condition, realize $\psi$ with negative values.)

\[
\psi_0(t) = \psi_0, \quad \ddot{\psi} = \psi_0 + R_V(\lambda \dot{\varphi}^2 \psi), \quad R_V(t, s) = -(t - s)\vartheta(t - s).
\]

A \textbf{second order} estimate on the Minkowski vacuum gives

\[
\omega(\psi(f)) \approx \psi_0, \\
\varsigma^2(f) \approx \omega(\psi_1(f) \ast \omega \psi_1(f)) = \frac{\psi_0^2}{\pi^2 7!} \int_{0}^{+\infty} dp \ p^3 \hat{f}(p) \hat{f}(p).
\]

- $f$ is a smooth approximation of the characteristic function of the time interval $I$.

- The \textbf{smaller} the support, the \textbf{larger} the variance.
The probability density of $\psi$ is approximated by a Gaussian distribution

$$\mathbb{P}(\psi(f_\tau) \leq 0) \approx N(-\psi_0, 0, 1), \quad f_\tau(s) := f(s - \tau).$$

Consider a sequence $\{X_n\}_n$ of random variables such that

$$X_n \sim \psi(f_\tau) \quad \forall n,$$

- Focusing occurs.
- **Time of the first collapse** is distributed as an exponential of parameter $\lambda_\tau := \mathbb{P}(\psi(f_\tau) \leq 0)$.
- The result is qualitatively similar to the one obtained by Carlip et al.
- The larger the support of $f$ the smaller the collapse probability due to quantum fluctuations.
Towards quantum spacetime?

- In [DFR 95] the authors find the commutation rules among the coordinates

\[ [q^\mu, q^\nu] = iQ^{\mu\nu} \]

compatible with the following uncertainty relations

\[ \Delta x_0 (\Delta x_1 + \Delta x_2 + \Delta x_3) \geq \lambda_P^2, \]
\[ \Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \geq \lambda_P^2 \]

which are obtained using the following:

**Minimal Principle:**

We cannot create a singularity just observing a system.

- Together with the **Heisenberg principle (HP)** (valid in Minkowski).
- The uncertainties are tailored to the flat spacetime.
In [Doplicher Morsella np 2013] the semiclassical equation in connection with that principle was used to obtain a minimal length scale in spherically symmetric spacetimes.

A model for a measuring apparatus was discussed and the preparation of the system was considered → kinematical point of view.

In the semiclassical approximation, the matter fluctuations can induce the formation of singularities.

They can be made small smearing over long time intervals.

Open task: Obtain bounds for the coordinate uncertainties relations without studying the measuring apparatus.
Summary

- Algebra of matter fields on timelike geodesics can be considered.

- Passive influence of matter fluctuation on expansion parameter can be studied within pAQFT.

- Bounds for uncertainty relations among spacetime coordinates can be studied.

Open Questions

- Can we get bounds for the validity of semiclassical equations?
- Can we do better than perturbation theory?
- Can we address intrinsic fluctuation of the expansion parameter?
- What about their influence on the matter?
- Quantum gravity solves those issues?
Thanks a lot for your attention!