Higher symmetries of Laplace and Dirac operators – towards supersymmetries

Jean-Philippe Michel
(Université de Liège, PAI DYGEST)

Fundamental Frontiers in Physics 2014
Marseille, 17th July

Definition of higher symmetries

Let $D \in \mathcal{D}(M)$ be a differential operator over M, e.g. $D = \Delta$ on $\mathbb{R}^n$.

Three types of symmetries:

- $X \in \mathrm{Vect}(M)$ s.t. $[D, X] = 0$, Lie algebra / preserve eigenspaces;
- $A \in \mathcal{D}(M)$ s.t. $[D, A] = 0$, associative algebra / preserve eigenspaces;
  $\rightsquigarrow$ commuting symmetries,
- $A \in \mathcal{D}(M)$ s.t. $D \circ A = B \circ D$, associative algebra / preserve $\ker D$;
  $\rightsquigarrow$ Higher Symmetries, trivial HS if $A = A_0 \circ D$.

Remark: generalize to $D \in \mathcal{D}(M, E)$ and $D \in \mathcal{D}(M; E, F)$.

Short review and aim

Higher symmetries of the Laplacian:

- symmetries of order 2 $\longleftrightarrow$ separation of variables of $\Delta\phi = E\phi$,
  - HS of order 2 classified in the flat case [Boyer, Kalnins, Miller '76],
  - commuting symmetries of order 2 in the curved case [Carter '77],
- full algebra of HS determined in the flat case [Eastwood '05],
- application in higher spin field theory [Vasiliev et al.].

HS of other differential operators have been investigated: Δ^k, $\not{D}$, ...

Aim of the talk:

- to present Eastwood's result via quantization methods,
- extension to the Dirac operator and to the system $\Delta \oplus \not{D}$.

Dequantization and symbol space

Order: locally, D is of order k if $D = D^{i_1\cdots i_k}(x)\partial_{i_1}\cdots\partial_{i_k} + \text{l.o.t.}$.
$\rightsquigarrow$ Increasing filtration: $\mathcal{D}(M) = \bigcup \mathcal{D}_k(M)$ with $\mathcal{D}_k(M)\cdot\mathcal{D}_l(M) \subseteq \mathcal{D}_{k+l}(M)$.

$\mathcal{D}(M)$ is an associative filtered algebra

Symbols: $\mathcal{S}(M) = \bigoplus \mathcal{S}_k(M)$ and
$\mathcal{S}_k(M) := \mathcal{D}_k(M)/\mathcal{D}_{k-1}(M) \cong \Gamma(\mathcal{S}^k TM) \cong \mathrm{Pol}_k(T^*M)$.

$$\sigma_k : D \mapsto \sigma_k(D) := D^{i_1\cdots i_k}(x)\partial_{i_1}\odot\cdots\odot\partial_{i_k} = D^{i_1\cdots i_k}(x)p_{i_1}\cdots p_{i_k}$$

Properties: if $\{\cdot,\cdot\} = \partial_{p_i}\otimes\partial_{x^i} - \partial_{x^i}\otimes\partial_{p_i}$ the Poisson bracket on T^*M, then

$$\sigma_{k+l}(A\circ B) = \sigma_k(A)\sigma_l(B),$$
$$\sigma_{k+l-1}([A,B]) = \{\sigma_k(A),\sigma_l(B)\}.$$

$\mathcal{S}(M)$ is a graded Poisson algebra

The Laplacian, its symbol and symmetries

Definition: on (M, g), we set $\Delta := \nabla_i g^{ij} \nabla_j + cR$, with $c \in \mathbb{R}$ and R the scalar curvature.

Symbol: $H = \sigma_2(\Delta) = g^{ij}\partial_i \odot \partial_j = g^{ij} p_i p_j$.

$$\text{Hamiltonian flow of } H \text{ on } T^*M \stackrel{g}{\longleftrightarrow} \text{geodesic spray on } TM$$

Symmetries:

- $[\Delta, X] = 0 \Rightarrow \{H, \sigma_1(X)\} = 0 = L_X g$, $\quad X$ Killing vector field;
- $[\Delta, A] = 0 \Rightarrow \{H, \sigma_k(A)\} = 0$, $\quad \sigma_k(A)$ Killing tensor field;
  $\rightsquigarrow$ hidden symmetry, i.e. symmetry of (T^*M, H) not of (M, g);
- $\Delta \circ A = B \circ \Delta \Rightarrow \{H, \sigma_k(A)\} \in (H)$, $\quad \sigma_k(A)$ conformal Killing tensor field;
  $\rightsquigarrow$ if $A = X + f \in \mathcal{D}_1(M)$, then $L_X g = F g$.

Concept of quantization

Definition: a quantization is a linear isomorphism $\mathcal{Q} : \mathcal{S}(M) \to \mathcal{D}(M)$, such that $\sigma_k \circ \mathcal{Q}(P) = P$ for all $P \in \mathcal{S}_k(M)$.

Facts:

- no quantization s.t. $[\mathcal{Q}(P_1), \mathcal{Q}(P_2)] = \mathcal{Q}(\{P_1, P_2\})$ for all $P_1, P_2 \in \mathcal{S}(M)$,
- if $[\mathcal{Q}(X), \mathcal{Q}(Y)] = \mathcal{Q}(\{X, Y\})$ for all $X, Y \in \mathcal{S}_1(M)$, then $\mathcal{Q}(X) = L_X = X + \lambda \mathrm{div} X$ is the Lie derivative on λ-densities, $\lambda \in \mathbb{R}$,
- no quantization s.t. $[\mathcal{Q}(X), \mathcal{Q}(P)] = \mathcal{Q}(\{X, P\})$ for all $X \in \mathcal{S}_1(M)$, $P \in \mathcal{S}(M)$.

Examples on $T^*\mathbb{R}^n$:

- normal ordering $\mathcal{N} : P^{i_1 \cdots i_k}(x) p_{i_1} \cdots p_{i_k} \mapsto P^{i_1 \cdots i_k}(x) \partial_{i_1} \cdots \partial_{i_k}$, $\mathcal{N}(X) = L_X$ on 0-densities, equiv. under $\mathfrak{gl}(n) \ltimes \mathfrak{heis}(2n) \cong \langle 1, x^i, p_i, x^i p_j \rangle$;
- Weyl quantization $\mathcal{Q}_W = \mathcal{N} \circ \exp\left(\frac{1}{2} \mathrm{div}\right)$, with $\mathrm{div} = \partial_i \partial_{p_i}$ the divergence, $\mathcal{Q}_W(X) = L_X$ on $\frac{1}{2}$-densities, equiv. under $\mathfrak{sp}(2n) \ltimes \mathfrak{heis}(2n) \cong \langle 1, x^i, p_i, x^i x^j, x^i p_j, p_i p_j \rangle$.

Symmetries of Δ on $\mathbb{R}^{p,q}$ (I)

Commuting symmetries of Δ :

- $[\Delta, \mathcal{Q}_W(P)] = 0 \Leftrightarrow \{H, P\} = 0$;
- $\{\text{Killing tensor field}\} \cong \mathrm{span}(\mathfrak{e}(p,q))$ in $\mathcal{S}(\mathbb{R}^n)$, with $\mathfrak{e}(p,q)$ the Lie algebra of Killing vector fields.

HS of order 1 of Δ :

- $L_X g = F g \Rightarrow \Delta \circ (X^i \partial_i + \frac{n-2}{2n} \partial_i X^i) = (X^i \partial_i + \frac{n+2}{2n} \partial_i X^i) \circ \Delta$, hence $\Delta : \Gamma(\mathrm{Vol}^{\otimes \frac{n-2}{2n}}) \to \Gamma(\mathrm{Vol}^{\otimes \frac{n+2}{2n}})$ is conformally invariant.
- $\mathfrak{conf}(\mathbb{R}^{p,q}) = \mathfrak{o}(p+1, q+1) \cong \langle p_i, x_i p_j - x_j p_i, x_i x^j p_j - \frac{1}{2} x^2 p_i \rangle$.

Theorem (Duval-Lecomte-Ovsienko, '99)

there exists a unique linear bijection

$$\mathcal{Q} : \mathcal{S}(\mathbb{R}^{p,q}) \to \mathcal{D}(\mathbb{R}^{p,q})$$

which is $\mathfrak{conf}(\mathbb{R}^{p,q})$-equivariant and compatible with principal symbol maps.

Symmetries of Δ on $\mathbb{R}^{p,q}$ (I)

Commuting symmetries of Δ :

- $[\Delta, \mathcal{Q}_W(P)] = 0 \Leftrightarrow \{H, P\} = 0$;
- {Killing tensor field} $\cong \mathrm{span}(\mathfrak{e}(p,q))$ in $\mathcal{S}(\mathbb{R}^n)$, with $\mathfrak{e}(p,q)$ the Lie algebra of Killing vector fields.

HS of order 1 of Δ :

- $L_X g = Fg \Rightarrow \Delta \circ (X^i\partial_i + \frac{n-2}{2n}\partial_i X^i) = (X^i\partial_i + \frac{n+2}{2n}\partial_i X^i) \circ \Delta$, hence $\Delta : \Gamma(\mathrm{Vol}^{\otimes \frac{n-2}{2n}}) \to \Gamma(\mathrm{Vol}^{\otimes \frac{n+2}{2n}})$ is conformally invariant.
- $\mathfrak{conf}(\mathbb{R}^{p,q}) = \mathfrak{o}(p+1, q+1) \cong \langle p_i, x_i p_j - x_j p_i, x_i x^j p_j - \frac{1}{2}x^2 p_i \rangle$.

Theorem (Duval-Lecomte-Ovsienko, '99)

For λ, μ generic there exists a unique linear bijection

$$\mathcal{Q}_{\lambda,\mu} : \mathcal{S}(\mathbb{R}^{p,q}) \otimes \Gamma(\mathrm{Vol}^{\otimes(\mu-\lambda)}) \to \mathcal{D}(\mathbb{R}^{p,q}; \mathrm{Vol}^{\otimes\lambda}, \mathrm{Vol}^{\otimes\mu})$$

which is $\mathfrak{conf}(\mathbb{R}^{p,q})$-equivariant and compatible with principal symbol maps.

Symmetries of Δ on $\mathbb{R}^{p,q}$ (II)

Theorem (Eastwood '05, M.'14)

- $\mathcal{Q}_{\frac{n-2}{2n},\frac{n-2}{2n}} : \{CK\text{-tensors}\} \xrightarrow{\sim} \{HS \text{ of } \Delta\}$.
- *$\{HS \text{ of } \Delta\}/\{\text{Trivial } HS\} \cong \mathfrak{U}(\mathfrak{g})/\mathcal{J}$, with $\mathfrak{g} = \mathfrak{o}(p+q+2,\mathbb{C})$ and $\mathcal{J}$ its Joseph ideal. Unique invariant star-deformation of $R[\overline{\mathcal{O}_{min}}]$.*

Remark: Let $\Delta_Y = \nabla_i g^{ij}\nabla_j + \frac{n-2}{4(n-1)}R$ on $\mathbb{S}^p \times \mathbb{S}^q$. If $p+q$ is even and ≥ 6, $\ker \Delta_Y$ is the minimal representation of $\mathrm{O}(p+1,q+1)$, and $\mathcal{J}$ is its vanishing ideal [Kobayashi-Orsted-..., '03-...].

Higher symmetries of Dirac operator on $\mathbb{R}^{p,q}$

Examples: first order HS are classified [Benn-Kress, '04]:

$$g^{ij}\gamma(\iota_{e_i}K)\nabla_{e_j} - \frac{\kappa}{\kappa+1}\gamma(\boldsymbol{d}K) + \frac{n-\kappa}{2(n+1-\kappa)}\gamma(\delta K)$$

where K belongs to $\{$conformal Killing κ-form$\} \cong \bigwedge^{\kappa+1}\mathbb{C}^{p+q+2}$.

Same strategy as for Δ applies:

- symbol space is a graded Poisson superalgebra ($\cong \Gamma(\mathcal{S}TM \otimes \bigwedge T^*M)$),
- $\not{D} : \Gamma(S \otimes \mathrm{Vol}^{\otimes \frac{n-1}{2n}}) \to \Gamma(S \otimes \mathrm{Vol}^{\otimes \frac{n+1}{2n}})$ is conformally invariant,
- conformally equivariant quantization exists and is unique [M., '09].

Theorem (Eastwwod-Somberg-Souček; M., Silhan)

- $\mathcal{Q}_{\frac{n-1}{2n},\frac{n-1}{2n}} : \{\textit{CK hook-tensors}\} \xrightarrow{\sim} \{\textit{HS of } \not{D}\}$.
- $\{\textit{HS of } \not{D}\}/\{\textit{Trivial HS}\} \cong \mathfrak{U}(\mathfrak{g})/\mathcal{J}_{\not{D}}$, *if $p+q$ is odd or $p+q$ is even and* $\not{D} : S^+ \to S^-$.

Problematic

Determine the algebra of HS of the system of differential operators

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^{p,q}) \oplus \Gamma(S) & \to & \mathcal{C}^\infty(\mathbb{R}^{p,q}) \oplus \Gamma(S) \\ \begin{pmatrix} f \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} \Delta f \\ \not{D}\phi \end{pmatrix} \end{array}$$

Problematic

Determine the algebra of HS of the system of differential operators

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^{p,q}) \oplus \Gamma(S) & \to & \mathcal{C}^\infty(\mathbb{R}^{p,q}) \oplus \Gamma(S) \\ \begin{pmatrix} f \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} \Delta f \\ \not{D}\phi \end{pmatrix} \end{array}$$

The HS read as

$$\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix} \begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix} = \begin{pmatrix} b & \beta^+ \\ \beta^- & B \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$$

with new symmetries:

$$\Delta\alpha^- = \beta^+\not{D} \quad \text{on } \Gamma(S) \qquad \text{and} \qquad \not{D}\alpha^+ = \beta^-\Delta \quad \text{on } \mathcal{C}^\infty(\mathbb{R}^{p,q}).$$

Classification of HS

Examples: if $\Lambda \in \Gamma(S)$ is a twistor spinor, $\nabla_i \Lambda = -\frac{1}{n}\gamma_i(\not{D}\Lambda)$, the following are symmetries [Wess-Zumino, Nucl. Phys. B 1974]:

$$\Delta \alpha_\Lambda^- = \beta_\Lambda^+ \not{D}, \quad \begin{cases} \alpha_\Lambda^-(\phi) = \varepsilon(\Lambda, \phi), \\ \beta_\Lambda^+(\not{D}\phi) = \varepsilon(\Lambda, \not{D}^2\phi) + \frac{2}{n}\varepsilon(\not{D}\Lambda, \not{D}\phi), \end{cases} \quad \phi \in \Gamma(S);$$

$$\not{D}\alpha_\Lambda^+ = \beta_\Lambda^- \Delta, \quad \begin{cases} \alpha_\Lambda^+(f) = \gamma^i(\Lambda)\partial_i f + \frac{n-2}{n}(\not{D}\Lambda)\cdot f, \\ \beta_\Lambda^-(\Delta f) = \Lambda \cdot \Delta f, \end{cases} \quad f \in \mathcal{C}^\infty(\mathbb{R}^{p,q}).$$

Proposition (M., Silhan)

The matrix of operators $\begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix}$ *is a HS iff*

- *a is a HS of Δ and A is a HS of $\not{D}$,*
- *$\alpha^- = \sum_i a_i \circ \alpha_{\Lambda_i}^-$, with a_i HS of Δ and $\alpha_{\Lambda_i}^-$ as above,*
- *$\alpha^+ = \sum_i \alpha_{\Lambda_i}^+ \circ a_i$, with a_i HS of Δ and $\alpha_{\Lambda_i}^+$ as above.*

Composition of twistor spinors actions

Lie (super-)algebra ?

Candidate: Conf. Killing vector fields $\oplus$ Twistor-spinors.

Hint from Rep. Theory:

- in odd dimension, $\mathrm{TwSp} \otimes \mathrm{TwSp} \cong \wedge^+ \mathbb{C}^{n+2} \cong$ space of conf. Killing odd forms;
- in even dimension, $\mathrm{TwSp} \otimes \mathrm{TwSp} \cong \wedge \mathbb{C}^{n+2} \cong$ space of all conf. Killing forms.

Fact: the composition $\alpha^+_{\Lambda'} \circ \alpha^-_{\Lambda}$ gives indeed rise to all HS of 1st order of $\not{D}$.

Proposition (M., Silhan)

If $n \geq 5$, there exists no Lie (super-)algebra containing $\mathfrak{o}(n+2,\mathbb{C})$ and generating the algebra of HS of $\Delta \oplus \not{D}$.

Supergeometric reformulation in dimension 3

$\Pi S^* \cong \mathbb{R}^{3|2}$ is a supermanifold with sheaf of functions

$$\mathcal{O}(\Pi S^*) = \mathcal{C}^\infty(\mathbb{R}^3) \oplus \Gamma(S) \oplus \Gamma(\wedge^2 S).$$

The pairing ε is a distinguished element of $\Gamma(\wedge^2 S)$. We define

$$\boxslash : \mathcal{O}\Big(\Pi S^*\Big) \to \mathcal{O}\Big(\Pi S^*\Big)$$

by the formula $\boxslash := \varepsilon\Delta + \not{D} + \varepsilon^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \not{D} & 0 \\ \Delta & 0 & 0 \end{pmatrix}$.

Proposition

We have the isomorphism of algebras

$$\{HS\ of\ \boxslash\}/\{trivial\ HS\} \cong \{HS\ of \begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}\}/\{trivial\ HS\}.$$

Twistor-spinors as odd vector fields

Let (x^i, θ^a) be coordinates on $\mathbb{R}^{3|2}$ and $(\partial_i, \partial_{\theta^a})$ the corresponding derivatives.

For all $\lambda \in \mathbb{R}$, we define on $\mathcal{O}\Big(\Pi S^*\Big)$

- $L_X = X^i \partial_i - \frac{1}{2}\gamma(\boldsymbol{d}X^\flat)_a^b \theta^a \partial_{\theta^b} + (\lambda + \frac{1}{2n}\theta^a \partial_{\theta^a})(\partial_i X^i)$,
- $L_\Lambda^+ = \gamma^i{}_b^a \Lambda_a \theta^b \partial_i + 2(\lambda - \frac{1}{n}\theta^a \partial_{\theta^a})(\not{D}\Lambda)$,
- $L_\Lambda^- = \varepsilon^{ab}\Lambda_a \partial_{\theta^b}$.

Proposition

The space $\langle c, L_X \rangle \oplus \langle L_\Lambda^+, L_\Lambda^- \rangle$ *is stable under the commutator in* $\mathcal{D}(\mathbb{R}^{3|2})$ *and isomorphic to the Lie superalgebra* $\mathrm{spo}(4|2)$.

For $\lambda = \frac{n-2}{2n}$, we have

$$\boxslash L_X = L_X \boxslash, \qquad \boxslash L_\Lambda^+ = L_\Lambda^- \boxslash, \qquad \boxslash L_\Lambda^- = L_\Lambda^+ \boxslash\,.$$

Main result

Theorem (M., Šilhan)

In dim 3, $\{HS\ of\ \Delta \oplus \not{D}\}/\{Trivial\ HS\} \cong \mathfrak{U}(\mathfrak{spo}(4|2)/\mathcal{J}$, with $\mathcal{J}$ a Joseph-like ideal.

In dim 4, $\{HS\ of\ \Delta \oplus \not{D}\}/\{Trivial\ HS\} \cong \mathfrak{U}(\mathfrak{sl}(4|1)/\mathcal{J}$.