Spectral geometry with a cut-off

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High momentum/short distance cut-off Λ to regularize quantum field theory.
In Connes spectral action, one cuts-off the spectrum of the Dirac operator:

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\, f\left(\frac{D}{\Lambda}\right)$$

where f is the characteristic function of the interval $[0,1]$.

- What about truncating the Dirac operator from the beginning ?

$$D \to D_\Lambda := P_\Lambda\, D\, P_\Lambda$$

  with P_Λ a projection, for instance on the eigenspaces of D.

- What is the effect of this cut-off on the geometry ?
- Does it make a minimal length emerge (e.g. by the NCG distance formula) ?

Spectral geometry with a cut-off, F. D'Andrea, PM, F. Lizzi, J. Geo. Phy. (2014).

Outline:

1. Metric aspect of noncommutative geometry

2. Cutting-off the geometry
   - minimal length on a manifold
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1. The metric aspect of noncommutative geometry

- A point of a locally compact topological space $\mathcal{X}$ is a pure state of the C^*-commutative algebra $C_0(\mathcal{X})$:

$$\mathcal{X} \simeq \mathcal{P}(C_0(\mathcal{X})).$$

- Any commutative C^*-algebra comes in this way: Gelfand duality

$$\mathcal{A} \simeq C_0(\mathcal{P}(\mathcal{A})).$$

Spectral triple: Involutive algebra $\mathcal{A}$, faithful representation π on $\mathcal{H}$, operator D on $\mathcal{H}$ such that $[D, \pi(a)]$ is bounded and $\pi(a)[D - \lambda\mathbb{I}]^{-1}$ is compact for any $a \in \mathcal{A}$ and $\lambda \notin$ Sp D. When a set of conditions is satisfied, then

Theorem Connes 1996-2008

$\mathcal{M}$ a compact Riemann spin manifold, then $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/ = -i\gamma^\mu\partial_\mu)$ is a spectral triple.

When $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

Given any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A}$ commutative or not, one defines on its state space $\mathcal{S}(\mathcal{A})$ the spectral distance (possibly infinite)

$$d_D(\varphi, \psi) := \sup_{a \in \mathcal{A}} \{|\varphi(a) - \psi(a)|, L_D(a) \leq 1\}$$

where L_D is the seminorm

$$L_D(a) := \|[D, \pi(a)]\| \, .$$

- coherent with the classical case when $\mathcal{A} = C_0^\infty(\mathcal{M})$: $d_{\partial\!\!\!/} = d_{geo}$;
- does not involve notions ill-defined in a quantum context (e.g. trajectories between points) but only spectral properties $\Longrightarrow$ suitable for cutting-off the geometry;
- Noncommutative generalization of the Wasserstein distance in the theory of optimal transport.

2. Cutting-off the geometry

On manifold, replace D with D_Λ such that $||D_\Lambda|| = \Lambda$.

Proposition D'Andrea, Lizzi, P.M. 2013

$$d_{D_\Lambda}(\delta_x, \delta_y) \geq \Lambda^{-1}.$$

In particular if D_Λ is a finite rank operator, then $d_{D_\Lambda}(\delta_x, \delta_y) = \infty$

Cutting-off all but the first N Fourier modes ($D_\Lambda = D_N := P_N D P_N$) destroys the metric structure. Need to approximate points by non-pure states, e.g. on S^1 by the Fejer transform

$$\Psi_{x,N}(f) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N+1}) f_n e^{inx}.$$

Proposition D'Andrea, Lizzi, P.M. 2013

$$d_{D_N}(\Psi_{x,N}, \Psi_{y,N}) \leq d_{\mathrm{geo}}(x, y),$$

$$\lim_{N\to\infty} d_{D_N}(\Psi_{x,N}, \Psi_{y,N}) = d_{\mathrm{geo}}(x, y) \qquad \forall x, y \in S^1.$$

Topologies induced by finite rank truncations

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, P_N a finite rank projection in $\mathcal{B}(\mathcal{H})$ and

$$\mathcal{O}_N := P_N \, \pi(\mathcal{A}_{sa}) \, P_N.$$

Three distances

$$\big(\mathcal{S}(\mathcal{A}), d_{\mathcal{A},D}\big) \qquad \big(\mathcal{S}(\mathcal{A}), d_{\mathcal{A},D_N}\big) \qquad \big(\mathcal{S}(\mathcal{O}_N), d_{\mathcal{O}_N,D_N}\big).$$

The map $\varphi^\sharp := \varphi \circ \mathrm{Ad}\, P_N$ sends a state φ of $\mathcal{O}_N$ to a normal state $\varphi^\sharp$ of $\pi(\mathcal{A})$.

$$d^\flat_{\mathcal{A},D}(\varphi,\psi) := d_{\mathcal{A},D}(\varphi^\sharp,\psi^\sharp), \quad d^\flat_{\mathcal{A},D_N}(\varphi,\psi) := d_{\mathcal{A},D_N}(\varphi^\sharp,\psi^\sharp).$$

Proposition

D'Andrea, Lizzi, P.M. 2013

If $L_N(a) := ||[D_N, a]|| = 0 \iff a = \mathbb{C}\mathbf{1}$ (Lipschitz condition) then

$$d^\flat_{\mathcal{A},D_N} \overset{strong}{\simeq} d_{\mathcal{O}_N,D_N}.$$

If in addition $i)$ L_D is Lipschitz, or $ii)$ $P_N \in \mathcal{A}'$, or $iii)$ $[D, P_N] = 0$, then

$$d^\flat_{\mathcal{A},D} \overset{strong}{\simeq} d_{\mathcal{O}_N,D_N}.$$

3. Convergence of truncations

$\{P_N\}_{N\in\mathbb{N}}$ a sequence of increasing finite-rank projections, weakly converging to 1.

Proposition D'Andrea, Lizzi, P.M.

For any normal state $\varphi \in \mathcal{N}(\mathcal{A})$, there is a sequence $\{\varphi_N\} \in \mathcal{S}(\mathcal{O}_N)$ such that $\varphi_N^\sharp \to \varphi$ in the weak* topology.

$(\mathcal{A}, L_D)$ is a compact quantum metric space iff $\mathcal{A}$ is unital, $L_D = ||[D,\cdot]||$ is Lipschitz and the spectral distance induced on $\mathcal{S}(\mathcal{A})$ the weak* topology. Rieffel

Proposition D'Andrea, Lizzi, P.M.

Let $(\mathcal{A}, L_D)$ be a compact quantum metric space. Then $(\mathcal{S}(\mathcal{O}_N), d^\flat_{\mathcal{A},D})$ converges to $(\overline{\mathcal{N}(\mathcal{A})}, d_{\mathcal{A},D})$ for the Gromov-Hausdorff distance.

Beyond compact quantum metric spaces

Commutative case

$\varphi \in \mathcal{S}(C_0^\infty(\mathcal{M})) \Longleftrightarrow$ a unique probability measure μ on $\mathcal{P}(C_0^\infty(\mathcal{M})) \simeq \mathcal{M}$ s.t.

$$\varphi(f) = \int_{\mathcal{M}} f(x)\,\mathrm{d}\mu_x \quad \forall f \in C_0(\mathcal{M}).$$

- For $\mathcal{M}$ connected, the finiteness of the moment of order 1 of φ,

$$\mathcal{M}_1(\varphi, x') := \int_{\mathcal{M}} d_{\text{geo}}(x, x')\,\mathrm{d}\mu(x)\ .$$

  does not depend on the choice of x'.

- The spectral distance $d_{\not{\partial}}$ between states with finite moment of order 1 is finite.

Noncommutative case

Definition

D'Andrea, Lizzi, P.M. 2013

$(\mathcal{A}, \mathcal{H}, D)$ a spectral triple and φ normal with density matrix R diagonal in the orthonormal basis $\mathfrak{B} = \{\psi_n\}_{n\in\mathbb{N}}$,

$$\varphi(a) = \sum_{n\geq 0} p_n \, \Psi_n(a) \quad \forall a \in \mathcal{A}.$$

We call moment of order 1 of R with respect to an eigenbasis $\mathfrak{B}$ and to a state Ψ_k (induced by a vector $\psi_k \in \mathfrak{B}$)

$$\mathcal{M}_1(R, \mathfrak{B}, \Psi_k) := \sum_{n\geq 0} p_n \, d_{\mathcal{A},D}(\Psi_k, \Psi_n).$$

- For different basis $\mathfrak{B}$, $\mathfrak{B}'$, one may have

$$\mathcal{M}_1(R, \mathfrak{B}, \Psi_k) = \infty \quad \text{while} \quad \mathcal{M}_1(R, \mathfrak{B}', \Psi'_k) < \infty.$$

- Once fixed R and $\mathfrak{B}$, the finiteness of $\mathcal{M}_1(R, \mathfrak{B}, \Psi_k)$ does not depend on Ψ_k.
- We write $\mathcal{N}_0(\mathcal{A})$ the set of normal states for which there exists at least one density matrix R with an eigenbasis $\mathfrak{B} = \{\psi_n\}$ such that

$$\mathcal{M}_1(R, \mathfrak{B}, \Psi_n) < \infty$$

Proposition

D'Andrea, Lizzi, P.M.

$(\mathcal{A}, \mathcal{H}, D)$ an arbitrary spectral triple, $\{P_N\}_{N\in\mathbb{N}}$ an increasing sequence of projections weakly convergent to 1.

For any $\varphi \in \mathcal{N}_0(\mathcal{A})$ such that

$$\mathcal{M}_1(R, \mathfrak{B}, \Psi_n) < \infty$$

for an eigenbasis $\mathfrak{B}$ in which the P_N's are all diagonal, there exists a sequence $\varphi_N \in \mathcal{S}(\mathcal{O}_N)$ such that

$$\lim_{N\to\infty} d_{\mathcal{A},D}(\varphi, \varphi_N^\sharp) = 0.$$

- On a non compact-quantum-metric space, any $\varphi \in \mathcal{N}_0(\mathcal{A})$ can be approximated in the metric topology by a truncation φ_N;
- unlike compact quantum metric spaces, the truncating-projections depend on the state.

4. An unbounded example

$(\mathcal{A}, \mathcal{H}, D)$ a spectral triple, P a projection such that $P \cdot \mathrm{dom}(D) \subset \mathrm{dom}(D)$.

$$D_P := P\, D\, P.$$

is not assumed to be bounded. Let $\mathcal{A}_P$ be the algebra generated by the elements $\pi(a) := PaP$, with $a \in \mathcal{A}$.

Proposition

D'Andrea, Lizzi, P.M.

There is a map $\sharp : \mathcal{S}(\mathcal{A}_P) \to \mathcal{S}(\mathcal{A})$, $\varphi \mapsto \varphi^\sharp$, given by

$$\varphi^\sharp = \varphi \circ \pi \ .$$

If $[P, D] = 0$ or $[P, a] = 0$ for all $a \in \mathcal{A}$, then for any $\varphi, \psi \in \mathcal{S}(\mathcal{A}_P)$ one has

$$d_{\mathcal{A}_P, D_P}(\varphi, \psi) \geq d_{\mathcal{A}, D}(\varphi^\sharp, \psi^\sharp).$$

Berezin quantization of the plane

Let $\mathcal{H} = L^2(\mathbb{C}, \frac{d^2z}{\pi})$. Take

$$\mathcal{A} = \mathcal{S}(\mathbb{R}^2),\ \mathcal{H} \otimes \mathbb{C}^2,\ D = \sigma^\mu \partial_\mu.$$

For $\theta > 0$ ($\theta = \lambda_p^2$), define

$$\mathcal{H}_\theta := \text{Span}\left\{ h_n(z) := \frac{z^n}{\sqrt{\theta^{n+1} n!}}\, e^{-\frac{|z|^2}{2\theta}} \right\}_{n\in\mathbb{N}} \qquad P_\theta : \mathcal{H} \to \mathcal{H}_\theta.$$

Let $\mathcal{O}_\theta$ be the order unit space generated by $\pi_\theta(f) := P_\theta\, f\, P_\theta$ (for $f = f^*$), and

$$D_\theta := (P_\theta \otimes \mathbb{I}_2)\, D\, (P_\theta \otimes \mathbb{I}_2) = \frac{2}{\sqrt{\theta}} \begin{pmatrix} 0 & \mathfrak{a}^\dagger \\ \mathfrak{a} & 0 \end{pmatrix}$$

with $\mathfrak{a}^\dagger, \mathfrak{a}$ are the creation, annihilation operators on the h_n's.

Berezin transform of f:

$$B_\theta(f) : z \to \langle \psi_z, P_\theta(f)\, \psi_z \rangle \quad \text{where} \quad \psi_z = e^{-\frac{|z|^2}{2\theta}} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{\theta^n n!}} h_n.$$

Definition

D'Andrea, Lizzi, P.M. 2013

For any states φ, ψ of $\mathcal{A}$, define

$$d^{(\theta)}_{\mathcal{A},D}(\varphi,\psi) := \sup_{f=f^*\in\mathcal{A}} \left\{\varphi(f) - \psi(f),\ ||[D, B^\theta(f)|| \leq 1\right\}$$

Proposition

D'Andrea, Lizzi, P.M. 2013

For all $\varphi, \psi \in \mathcal{S}(\mathcal{O}_\theta)$:

$$d_{\mathcal{A},D}(\varphi^\sharp, \psi^\sharp) \leq d_{\mathcal{O}_\theta, D_\theta}(\varphi,\psi) \leq d^{(\theta)}_{\mathcal{A},D}(\varphi^\sharp,\psi^\sharp).$$

In particular, the distance between coherent states $\Psi_z, \Psi_{z'}$, $z, z' \in \mathbb{C}$, is

$$d_{\mathcal{O}_\theta, D_\theta}(\Psi_z, \Psi_{z'}) = |z - z'|.$$

5. Quantizing the distance

Length operator on noncommutative spacetimes:

$$[q_\mu, q_\nu] = i\theta_{\mu\nu}\mathbb{I} \quad \Longrightarrow L \doteq \sqrt{\sum_{\mu=1}^{2N} dq_\mu^2} \quad \text{with} \quad dq_\mu = q_\mu \otimes \mathbb{I} - \mathbb{I} \otimes q_\mu,$$

q_μ affiliated to $\mathbb{K}$. Given $\varphi \otimes \psi \in \mathcal{S}(\mathbb{K})$, define the *quantum length* of $\varphi \otimes \psi$ as

$$d_L(\varphi, \psi) \doteq (\varphi \otimes \psi)(L).$$

Spectral triple of the Moyal plane:

$$\mathcal{A} = (S(\mathbb{R}^2), \star), \quad \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, \quad \partial\!\!\!/ = -i\sigma^\mu \partial_\mu$$

where $\star$ is a noncommutative product induced by a symplectic structure on $\mathbb{R}^2$.

- In the commutative case $d_L(\delta_x, \delta_y) = d_{\text{geo}}(x, y) = d_{\partial\!\!\!/}(\delta_x, \delta_y)$.
- In the noncommutative case $d_D(\varphi, \varphi) = 0$ while $d_L(\varphi, \varphi) = (\varphi \otimes \varphi)(L)$ has no reason to vanish !

$$\mathcal{A}' \doteq \mathcal{A} \otimes \mathbb{C}^2, \quad \mathcal{H}' \doteq \mathcal{H} \otimes \mathbb{C}^2, \quad D' \doteq \partial\!\!\!/ \otimes \mathbb{I} + \gamma^5 \otimes \begin{pmatrix} 0 & \Lambda^{-1} \\ \Lambda^{-1} & 0 \end{pmatrix} \text{ with } \Lambda = \text{ const.}$$

States: (φ, δ_i), with $\delta_i, i = 1, 2$, the pure states of $\mathbb{C}^2$.

Thanks to Pythagoras equality

generalized in D'Andrea, PM 2013

$$d_{D'}((\varphi, \delta_1), (\varphi', \delta_2)) = \sqrt{d^2_{\partial\!\!\!/}(\varphi, \varphi') + \Lambda^2},$$

one has that quantizing the spectral distance

$$\sqrt{d_{L^2}(\varphi, \varphi')} = d_{D'}((\varphi, \delta_1), (\varphi', \delta_2))$$

is equivalent to de-quantizing the quantum length

$$d_D(\varphi, \varphi') = d'_L(\varphi, \varphi') \doteq \sqrt{d_{L^2}(\varphi, \varphi') - \Lambda^2} \qquad (1)$$

for $\Lambda = \min(d_L(\varphi, \varphi), d_L(\varphi', \varphi'))$.

Proposition

P.M., L. Tomassini (2011)

Condition (1) holds true for translated states: $\varphi_\kappa(f) = \varphi(f(\cdot + \kappa))$ for any $\kappa \in \mathbb{C}$

$$d_D(\varphi, \varphi_\kappa) = \lambda_P \sqrt{2} |\kappa| = d'_L(\varphi, \varphi_\kappa).$$

Conclusion

- Truncating the Dirac operator on a manifold yields a non-zero minimal length between points, possibly infinite
  $\Longrightarrow$ approximation of points by non pure states.
- Other possibility to get a minimal length is to "quantize" Connes distance by doubling the spectral triple
  $\Longrightarrow$ yields the quantum length of the DFR model.
- Application in Berezin quantization using unbounded truncated Dirac operators.
- Topological properties of the space of truncated states: work well for compact quantum metric case, otherwise ask for the elaboration of a theory of optimal transport in noncommutative geometry.
- At the end of the day, the Higgs fields should appear as cost functions for the Monge problem on a noncommutative space.

Spectral geometry with a cut-off, with F. D'Andrea and F. Lizzi,
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On Pythagoras theorem for the product of spectral triples, with F. D'Andrea,
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Minimal length in quantum space & integrations of the line element in noncommutative geo., with F. Mercati and L. Tomassini,
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Noncommutative geometry of the Moyal plane: translation isometries, Connes distance between coherent states, Pythagoras equality, with L. Tomassini,
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1. Dimension: D^{-1} is an infinitesimal of order $\frac{1}{m}$.

2. Regularity: for any $a \in \mathcal{A}$, a and $[D, a]$ belong to the intersection of the domains of all the powers δ^k of the derivation $\delta(b) \doteq [|D|, b]$, where b belongs to the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$.

3. Finitude: $\mathcal{A}$ is a pre-C^*-algebra and the set $\mathcal{H}^\infty \doteq \bigcap_{k \in \mathbb{N}} \text{Dom } D^k$ of smooth vectors of $\mathcal{H}$ is a finite projective module.

4. First order: the representation of $\mathcal{A}^\circ$ commutes with $[D, \mathcal{A}]$

$$[[D, a], Jb^* J^{-1}] = 0 \text{ for all } a, b \in \mathcal{A}.$$

5. Orientability: there exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ such that $\pi(c) = \Gamma$.

6. Reality $(\mathcal{A} \otimes \mathcal{A}^\circ, \mathcal{H}, D, \Gamma, J)$ is aKR^n-cycle with $[a, Jb^* J^{-1}] = 0$. J is called the *real structure*. That is

- J is a anti-unitary bijection on $\mathcal{H}$ that implements the involution, i.e. $JaJ^{-1} = a^*$ for all $a \in \mathcal{A}$;
- if n is even, there is a graduation Γ of $\mathcal{H}$ that commutes with $\mathcal{A}$ and anticommutes with D;
- the following table holds

n mod 8	0	1	2	3	4	5	6	7
$J^2 = \pm\mathbb{I}$	+	+	-	-	-	-	+	+
$JD = \pm DJ$	+	-	+	+	+	-	+	+
$J\Gamma = \pm\Gamma J$	+		-		+		-	

For odd n, one sets $\Gamma = \mathbb{I}$.

7. Poincaré duality: the additive coupling on $K_*(\mathcal{A})$ coming from the index of the Dirac operator is non-degenerated.

Interlude: Monge-Kantorovich distance in optimal transport theory

$\mathcal{X}$ a locally compact Polish space, $c(x,y)$ a positive real function, the "cost".
The minimal work W required to transport the probability measure μ to ν is

$$W(\mu,\nu) \doteq \inf_{\pi} \int_{\mathcal{X}\times\mathcal{X}} c(x,y)\,\mathrm{d}\pi$$

with infimum on all the measures π on $\mathcal{X}\times\mathcal{X}$ with marginals μ,ν.

When the cost is a distance d, then W is a distance (possibly infinite) between probability measures on $\mathcal{X}$ (the Monge-Kantorovich or Wasserstein distance).

W is finite on the set of probability measures with finite moment of order 1,

$$\int_{\mathcal{X}} d(x_0,x)\mathrm{d}\mu(x) < +\infty \quad x_0 \in \mathcal{X}.$$

Proposition

Rieffel 99, then D'Andrea, P.M. 2009

Let $\mathcal{X} = \mathcal{M}$ be a complete, connected, without boundary, Riemannian manifold.
The distance of $\left(C_0^\infty(\mathcal{M}), L^2(\mathcal{M},S), \partial\!\!\!/\right)$ is the M.-K. distance with cost d_{geo} :

$$W(\mu,\nu) = d_{\partial\!\!\!/}(\varphi,\psi)$$

for any $\varphi,\psi \in \mathcal{S}(C_0(\mathcal{M}))$, with $\varphi(f) = \int_{\mathcal{M}} f\,\mathrm{d}\mu$ and $\psi(f) = \int_{\mathcal{M}} f\,\mathrm{d}\nu$.

Order unit space

A order unit space $\mathcal{O}$ is a partially ordered vector space with order unit e such that

- $\forall a \in \mathcal{O},$ there exists $r \in \mathbb{R}$ such that $a < re$,
- $a < re \quad \forall r > 0 \Longrightarrow a \leq 0.$

A norm is given by

$$||a|| = \inf\{r > 0,\ -re \leq a \leq re\}.$$

- Any real vector subspace $\mathcal{O}$ of $\mathcal{B}(\mathcal{H})_{sa}$ containing the identity 1 is an order unit space for the partial ordering of operators, with order unit $e = 1$.

Gromov-Hausdorff distance

X, Y subsets of a metric space (M, d).

$$d(x, Y) := \inf_{y \in Y} d(x, y)$$

$$d(X, Y) := \sup_{x \in X} d(x, Y)$$

Hausdorff distance between X and Y:

$$d_H(X, Y) := \max \left\{ d(X, Y), d(Y, X) \right\} .$$

Gromov-Hausdorff distance between two complete metric spaces $(X, d_X), (Y, d_Y)$:

$$d_{GH}(X, Y) := \inf d_H(f(x), g(y))$$

with infimum over all the isometric embeddings

$$f : X \to M, \quad g : Y \to M$$

into a metric space M.