# Noncommutative Geometry, the Spectral Action
# and Fundamental Symmetries

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In keeping with the title of this conference one may ask: *Where is the frontier of physics?*

Of course there are several answers, and the number of parallel session testifies this. But a frontier is a *a line of division between different or opposed things*. So we should put the frontier somewhere. **Were ?**

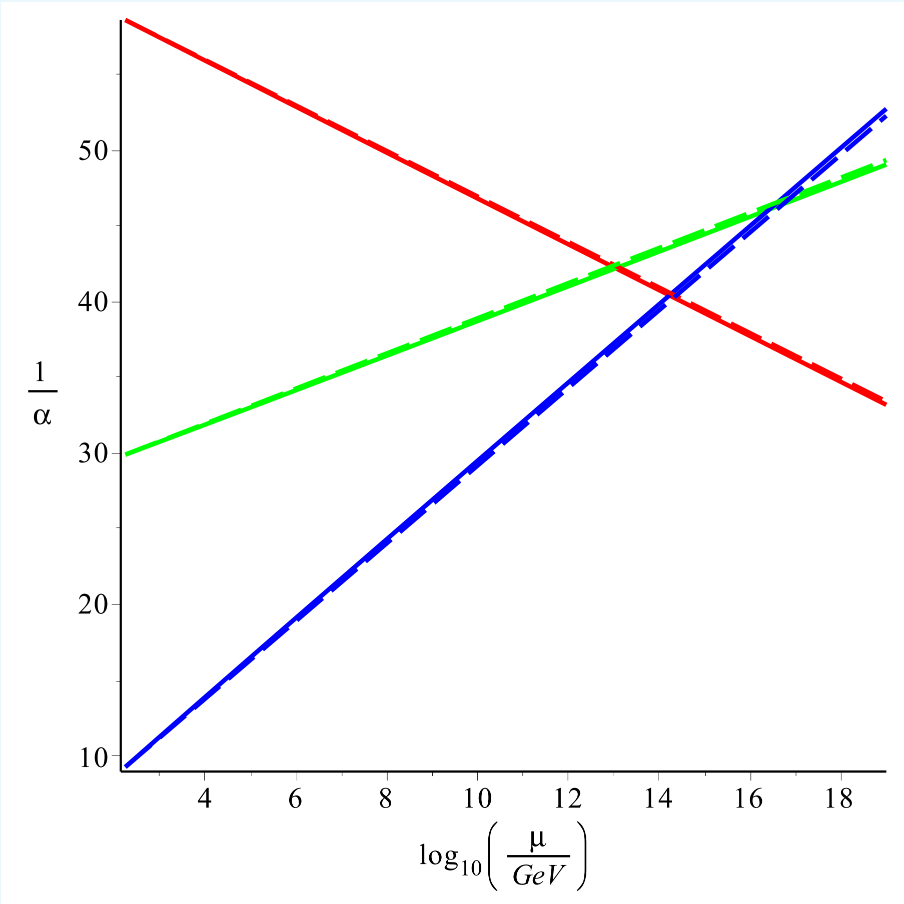
One natural frontier is of course the Planck scale. We know there we are in foreign territory. Gravity and quantum field theory are irreconcilables, we will have to use a new theory. But there can be something before.

We can use the knowledge form field theory at energies below the frontier to gather information

We are having new data from “high” energy experiments, mainly *LHC*, so this is a good moment to explore the consequences of field theory.

The way one can learn what happens beyond the scale of an experiment is to use the renormalization flow of the theory

We know that the coupling constants, i.e. the strength of the interaction, change with energy.



This picture is valid in the absence of new physics, i.e. new particles and new interactions which would alter the equations which govern the running

The three interaction strength start from rather different values but come together almost at a single unification point

But then the nonabelian interactions proceed towards asymptotic freedom, while the abelian one climbs towards a Landau pole at incredibly high energies $10^{53}$ GeV

The lack of a unification point was one of the reasons for the falling out of fashion of GUT's.

Some supersymmetric theories have unification point

Using the geographical analogy, we have known for a long time that the geometry we learn in high school, the one made of points, lines and surfaces, is a good vehicle to explore the world till we reached a frontier. Quantum land requires Noncommutative Geometry of Phase Space.

Therefore let us try to approach the frontier using noncommutative geometry

For the purpose of field theory, the novelty of noncommutative geometry is the fact that it is a **spectral** theory

One can put together what I say earlier about the presence of a frontier, and the need for a cutoff in field theory

There is a way to regularize the infinities of field theory based on the cutoff of the Dirac (or in general the wave) operator: Finite Mode regularization
Andrianov, Bonora, Fujikawa, FL, Kurkov

Without going into details, the cutoff is implemented by truncating the spectrum of the Dirac operator at a cutoff scale $\boxed{\Lambda}$. One can imagine then that at this scale a phase transition may take place.

The action then develops a scale anomaly, and the renormalization flow leads to the presence of the spectral action, which I will re-introduce below

Let me just mention that a study of the action beyond the cutoff scale indicates a space in which the correlations among point vanish, leading to a space for which "the points do not talk to each other" FL, Kurkov

The starting point of Connes' approach to is that geometry and its (noncommutative) generalizations are described by the spectral data of three basic ingredients:

- An algebra $\boxed{\mathcal{A}}$ which describes the topology of spacetime.

- A Hilbert space $\boxed{\mathcal{H}}$ on which the algebra act as operators, and which also describes the matter fields of the theory.

- A (generalized) Dirac Operator $\boxed{D_0}$ which carries all the information of the metric structure of the space, as well as other crucial information about the fermions.

An important role is also played by two other operators: the chirality $\boxed{\gamma}$ and charge conjugation $\boxed{J}$

There is a profound mathematical result (Gefand-Najmark) which states that the category of commmutative $\boxed{C^*}$-algebras and that of topological Hausdorff spaces are in one to one correspondence. The algebra being that of continuous complex valued functions on the space.

Connes programme is the transcription of all usual geometrical objects into algebraic terms, so to provide a ready generalization to the case for which the algebra is noncommutative

The points of the space (that can be reconstructed) are pure states, or maximal ideals of the algebra, or irreducible representations. They all coincide in the commutative case.

The geometric aspects are encoded in the Dirac operator.

In the commutative case it is possible to characterize a manifold with properties of the elements of the triple (all five of them)

There is a list of conditions and a theorem (Connes) which proves this.

Since the conditions are all purely algebraic there remain valid in the noncommutative case, defining a noncommutative manifold

In case you want to see them:

1. Dimension There is a nonnegative integer $n$ such that the eigenvalues of $D_0$ grow os $O(\frac{1}{n})$.

2. Regularity For any $a \in \mathcal{A}$ both $a$ and $[D_0, a]$ belong to the domain of $\delta^k$ for any integer $k$, where $\delta$ is the derivation given by $\delta(T) = [|D|, T]$.

3. Finiteness The space $\bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$ module.

4. Reality There exist $J$ with the commutation relation fixed by the number of dimensions with the property
   (a) Commutant $[a, Jb^* J^{-1}] = 0, \forall a, b$
   (b) First order $[[D, a], b^o = Jb^* J^{-1}] = 0 \ , \forall a, b$

5. Orientation There exists a Hochschild cycle $c$ of degree $n$ which gives the grading $\gamma$ , This condition gives an abstract volume form.

6. Poincaré duality A Certain intersection form detemrined by $D_0$ and by the K-theory of $\mathcal{A}$ and its opposite is nondegenrate.

While the formalism is geared towards the construction of genuine noncommutative spaces, spectacular, interesting results are obtained considering almost commutative geometries, which leads to: **Connes' approach to the standard model**

The project is to transcribe electrodynamics on an ordinary manifold using algebraic concepts: The algebra of functions, the Dirac operator, the Hilbert space and chirality and charge conjugation. One can then write the action in purely algebraic terms.

In this case the space is only "almost" noncommutative, in the sense that there still is an underlying spacetime, and and internal noncommutative but finite dimensional algebra

In these cases the algebra is of the kind $\boxed{\mathcal{A} = C(\mathbb{R}^4 \otimes \mathcal{A}_F)}$, where $\boxed{\mathcal{A}_F}$ is ma finite dimensional (matrix) algebra.

As Dirac operator: $D_0 = \partial\!\!\!/ + \psi\!\!\!/ \otimes \mathbb{I} + \gamma_5 \otimes D_F$

$D_F$ is a finite matrix containing masses (mixings) of the fermions

Its covariant version $D_A = D_0 + A + JAJ$, where $A$ is a one-form, we obtain the gauge vector bosons, and the Higgs boson which is like the internal component of the vector bosons

The spectral action is:

$$S_B = \operatorname{Tr} \chi \left( \frac{D_A}{\Lambda} \right)$$

where $\chi$ is a cutoff function, for example the characteristic function of the interval $[0, 1]$, in this case the action is just the number of eigenvalues of the Laplacian which are below the scale $\Lambda$

Then there is a "standard" fermionic action $\langle \Psi | D_A | \Psi \rangle$ which needs to be regularized, in the usual way (one can use the same cutoff)

In the work of Chamseddine, Connes and Marcolli the renormalization group flow is done by considering as boundary condition the unification of the three interaction coupling constants at $\boxed{\Lambda}$. This is approximately true.

The various couplings and parameters are then found at low energy via the renormalization flow

Yukawa couplings (masses) and mixings are taken as inputs. The mass parameter of the Higgs is however not needed, and is a function of the other parameters (which are dominated by the top mass).

There is therefore predictive power.

$\Lambda$ is the natural scale at which the theory lives, and is therefore the natural cutoff of the theory. Beyond such scale it is natural to think of the presence of a different theory.

In this sense the unification of the coupling constants, which is necessary for the theory, is not the prelude to another gauge theory with a larger unification group, but the signal of a new theory, of which the standard model is an effective theory. Later I will speculate more on this.

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of $\Lambda^{-1}$ as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the $f_n$ are the momenta of $\chi$

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the $a_n$ are the Seeley-de Witt coefficients which vanish for $n$ odd. For $D^2$ of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta)$$

Defining (in term of a generalized spin connection containing also the gauge fields)

$$\begin{aligned}\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right)\\ \Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]\\ E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)\end{aligned}$$

then

$$\begin{aligned}a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F\\ a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6}+E\right)\\ a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu}\\ &\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in $\Omega$ and $E$ are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of $D$

Take as finite algebra the one corresponding to the standard model, i.e. $\boxed{\mathcal{A}_F = \mathsf{Mat}(\mathbb{C})_3 \oplus \mathbb{H} \oplus \mathbb{C}}$

This algebra must be represented as operators on a Hilbert space, which also has a continuos infinite dimensional part (spinors on spacetime) times a finite dimensional one: $\boxed{\mathcal{H} = \mathsf{sp}(\mathbb{R}) \otimes \mathcal{H}_F}$. The grading given by $\boxed{\gamma}$ splits it into a left and right subspace: $\boxed{\mathcal{H}_L \oplus \mathcal{H}_R}$

The $\boxed{J}$ operator basically exchange the two chiralities and conjugates, thus effectively making the algebra act form the right.

For $\boxed{\mathcal{H}_{\mathcal{F}}}$ we take the zoo of known fermions: in total there are 96 degrees of freedom per generation, including right handed neutrinos, a relatively recent acquisition in the zoo.

Note the the full Hilbert space is the tensor product of this finite dimensional space times the usual spinorial degrees of freedom. So the states are overcounted. This is called fermion doubling.

One has to represent the algebra on this Hilbert space (in a reducible way) in such a way that restrictions imposed by the conditions shown earlyer are satisfied

The scheme does not work for all gauge theories. For example only representations of the algebra (not simply the group) are allowed. This means that only the fundamental and trivial representations are possible

True for the standard model, but not for the usual grand unified theories, like SU(5)

Then one cranks the machine and obtains the lagrangian of the standard model coupled with gravity

As I said the Dirac operator contains all data relative to the fermions, but no information on the Higgs mass (actually vev and quartic coupling coefficient) which can be calculated from the fermion mass parameters (Yukawa couplings). These in turn are dominated by the top quark coupling.

Hence we have a “prediction” for the Higgs mass.

The prediction is 170 GeV. The actual mass is 126 GeV.

If you take it as a mature fully formed theory then the result is wrong. Taking, as I do, it as a tool to investigate the standard model starting from first principles, then it is remarkable that a theory based on pure mathematical result gets reasonable numbers

Take the measurement of the Higgs as a reason to understand in which direction one has to improve on the theory

I will now try to understand in this framework the origin the standard model algebra, and see if it may shed light on the mass of the Higgs.

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}),$$

$\mathbb{H}$ are the quaternions, which we represent as $2 \times 2$ matrices

It is possible to have this emerge from the most general algebra which satisfies the condition of being a noncommutative manifold

The manifold conditions I flashed earlier are purly algebraic. Therefore they can be applied to finite dimensional (matrix) algebras. The resul is that only one kind of algebras are allowed:

$$\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}.$$

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$.
For a non trivial grading it must be $\boxed{a \geq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

You may have recognized before a Pati-Salam kind of symmetry. Its presence suggests the presence of a field which causes the breaking of o the standard model

This field, which we call $\boxed{\sigma}$ can in fact appear in the Dirac operator (Chamseddine and Connes), in the position corresponding to the neutrino Majorana mass

Doing again the running of the physical quantitates with this field does change the Higgs mass, making it compatible with the experimental value

Physics is therefore telling us that into his framework right handed neutrinos, and Majorana masses are crucial

Let us look in detail to a vector in the Hilbert space:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x) \in \mathcal{H} = L^2(\mathcal{M}) \otimes \mathsf{H}_F = sp(L^2(\mathcal{M})) \otimes \mathcal{H}_F$$

Note the difference between $\mathcal{H}_F$, which is 96 dimensional, and $\mathsf{H}_F$ which is 384 dimensional. The meaning of the indices is as follows:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$$\begin{array}{r} s = r, l \\ \dot{s} = \dot{0}, \dot{1} \end{array}$$

are the spinor indices. They are not internal indices in the sense that the algebra $\mathcal{A}_F$ acts diagonally on it. They take two values each, and together they make the four indices on an ordinary Dirac spinor.

$$\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\mathrm{I} = 0, \ldots 3$ indicates a "lepto-colour" index. The zeroth "colour" actually identifies leptons while $\mathrm{I} = 1, 2, 3$ are the usual three colours of QCD.

$$\Psi^{\mathrm{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\alpha = 1\ldots 4$ is the flavour index. It runs over the set $u_R, d_R, u_L, d_L$ when $\mathrm{I} = 1, 2, 3$, and $\nu_R, e_R, \nu_L, e_L$ when $\mathrm{I} = 0$. It repeats in the obvious way for the other generations.

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\mathsf{C} = 0, 1$ indicates whether we are considering "particles" ($\mathsf{C} = 0$) or "antiparticles" ($\mathsf{C} = 1$).

$$\boxed{\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)}$$

$\boxed{m = 1, 2, 3}$ is the generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters. For this seminar plays no role, and will ignored.

We can now give explicitly the algebra representations in term of these indices.

It splits between particles and antiparticles, with Quaternions acting on flavor indices, and complex numbers acting on lepto-colour. The presence of $\boxed{A + JAJ}$ provides the other action.

$\boxed{\gamma}$ matrices act on red indices, quaternions act on blue indices, complex matrices act on green indices

$$\boxed{\Psi^{\mathsf{C}\mathbf{I}m}_{s\dot{s}\alpha}(x)}$$

The representation of the other algebras use just subgroups. Details on demand

We can similarly write down the Dirac operator

$$\boxed{D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F}$$

$$\boxed{D_F = \begin{pmatrix} 0_{8N} & \mathcal{M} & \mathcal{M}_R & 0_{8N} \\ \mathcal{M}^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & \bar{\mathcal{M}} \\ 0_{8N} & 0_{8N} & \mathcal{M}^T & 0_{8N} \end{pmatrix}.}$$

$\boxed{\mathcal{M}}$ contains the Dirac-Yukawa couplings. It links left with right particles. $\boxed{\mathcal{M}_R = \mathcal{M}_R^T}$ contains Majorana masses and links righr particles with right antiparticles. $\boxed{\mathcal{M} = \begin{pmatrix} M_u & 0_{4N} \\ 0_{4N} & M_d \end{pmatrix} \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_{4N} \\ 0_{4N} & 0_{4N} \end{pmatrix}}$ where $\boxed{M_u}$ contaimins the masses of the up, charm and top quarks and the neutrinos (Dirac mass), $\boxed{M_R}$ contains the Majorana neutrinos masses and $\boxed{M_d}$ the remaining quarks and electrons, muon and tau masses, including mixings

Now you know the rules. With the algebra and $D$ one builds the one-form, which are the fluctuations of the Dirac operator.The bosonic fields are coming from these one-form $\sum_i a_i[D, b_i]$

But here we run into a problem: the elements of $\mathcal{M}_R$ are the ones which should give rise to the field $\sigma$ as intermediate boson, on a par with the Higgs, and relate to the breaking of the left-right symmetry.

Except that this term either commutes with $D$ or violates the first order condition!

One alternative would is to have a combination of algebra and Dirac operator violating the first order condition, i.e. the commutativity of function and potentials. This is the way later chosen by Chamseddine, Connes and Van Suijlekom

Or we may look for a bigger algebra...

Consider the case of $\boxed{\mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C})}$ for the case $\boxed{a = 4}$

In this case we need a $\boxed{2 \cdot (2 \cdot 4)^2 = 128}$ dimensional space, which for 3 generations gives a $\boxed{384}$ dimensional Hilbert space.

I need a representation of the algebra $\boxed{\mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C})}$ acting on the spinors I gave earlier, and the order zero conditions

I do not want to go into technical details (I could show slides with all indices in gory detail. . . ).

The fundamental point is that spinor indices and the internal gauge indices are mixed.

The two part of the algebra act on the indices like

$$\Psi^{\mathrm{c}\mathbf{I}m}_{s\dot{s}\alpha}(x)$$

Quaternions act on the blue indices, and complex numbers act on the red indices.

We are in a phase in which the Euclidean structure of space time has not yet emerged.

The fermions are not yet fermions

We envisage this Grand Symmetry to belong to a pre geometric phase. At this stage all elements of $D_F$ may be negligible, and the spinorial part of the direct operator $\partial\!\!\!/$ will cause the "breaking" to a phase in which the symmetries of the phase space emerge

In particular, the order one condition for $\partial\!\!\!/$ causes the reduction of the algebra to $\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$

And there is an added bonus:

This grand algebra, and a corresponding $\boxed{D}$ operator, have "more room" to operate. Although the Hilbert space is the same, we abandoned the factorization of the internal indices, giving us more entries to accommodate the Majorana masses

Hence we can put a Majorana mass for the neutrino and at the same time satisfy the order one condition. Then the one form corresponding to this $\boxed{D_\nu}$ will give us the by now famous field $\boxed{\sigma}$, which can only appear before the transition to the geometric spacetime. But we must abandon he boundedness of the algebra.

The natural scale for this mass is to be above a transition which gives the geometric structure. Therefore it is natural that it may be at a high scale. How high we can discuss

The grand symmetry is no ordinary gauge symmetry, there is never a $\boxed{SU(8)}$ in the game for example

It represents a phase in which the internal noncommutative geometry contains also the spin structure, even the Lorentz (Euclidean) structure of space time in a mixed way

The differentiation between the spin structure of spacetime, and the internal gauge theory comes as a breaking of the symmetry, triggered by $\boxed{\sigma}$, which now appears naturally has having to do with the geometry of spacetime.

What sort of spacetime do we have with this grand symmetry? Should we dare more and go non associative as well?