Hyperbolic PDEs with non-commutative time

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joint work with Rainer Verch

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The wave equation

Let's consider the wave operator on Minkowski space $\mathbb{R}^n$,

$$D = \frac{\partial^2}{\partial x_0^2} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2},$$

and the wave equation

$$Df = 0\,.$$

- Have Greens functions/fundamental solutions, i.e. linear cont. maps $R^\pm : \mathscr{C}_0^\infty \to \mathscr{C}^\infty$ such that

$$R^\pm Df = f = DR^\pm f, \qquad f \in \mathscr{C}_0^\infty\,.$$

- All solutions (with spacelike compact support) are of the form

$$R^- f - R^+ f, \qquad f \in \mathscr{C}_0^\infty\,.$$

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- Solutions spread with the speed of light $(= 1)$,

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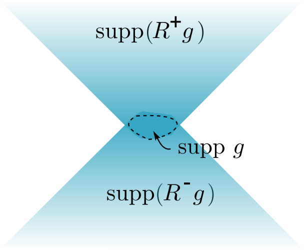

The wave equation

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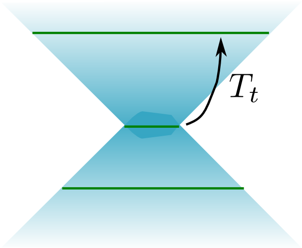
$$\mathrm{supp}(R^{\pm} g) \subset \mathcal{J}^{\pm}(\mathrm{supp}\, g)$$

- Cauchy problem well-posed, unique solutions to all $\mathscr{C}_0^\infty$-Cauchy data
- Have time evolution operator T_t



(Pre-)normally hyperbolic differential operators

All this is true for more general D; in particular for

- D normally hyperbolic,

$$D = \frac{\partial^2}{\partial x_0^2} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \sum_{\mu=0}^{n-1} U^\mu(x) \frac{\partial}{\partial x_\mu} + V(x),$$

- Or D "pre-normally hyperbolic" [Mühlhoff 2011]: D, D' first order diff. op. such that $D'D$ is normally hyperbolic (e.g. $D = -i\gamma^\mu \partial_\mu + V(x)$)

Perturbations

Consider (linear!) perturbations of D,

$$D_\lambda = D + \lambda \cdot W, \qquad \lambda \in \mathbb{C}\,.$$

- For example $(Wf)(x) = w(x) \cdot f(x)$ with $w \in \mathscr{C}_0^\infty$.
- Can study Cauchy problem or scattering problem (→ "relative Cauchy evolution" [Brunetti, Fredenhagen, Verch 2003])
- Møller operators

$$\Omega_{\lambda,\pm} : \mathrm{Sol}_\lambda \to \mathrm{Sol}_0$$

map "interacting" solution of D_λ to "free" solutions (future/past asymptotics)

- Scattering operator:

$$S_\lambda := \Omega_{\lambda,+}\Omega_{\lambda,-}^{-1} : \mathrm{Sol}_0 \to \mathrm{Sol}_0\,.$$

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Quantization

Since D is linear, corresponding field theory can be easily quantized (either in CCR or CAR fashion).

- For example for a Dirac operator, get Dirac quantum fields $\psi(f)$,

$$\psi(f)^*\psi(g) + \psi(g)\psi(f)^* = i\,\langle g, \gamma^0 Rf\rangle \cdot 1\,,$$

  CAR algebras $\mathfrak{F}_0, \mathfrak{F}_\lambda$, and

- a scattering automorphism

$$s_\lambda : \mathfrak{F}_0 \to \mathfrak{F}_0\,,$$

  induced by S_λ.

- Interesting quantity: Derivation ("Bogoliubov's formula")

$$\left.\frac{ds_\lambda(\psi(f))}{d\lambda}\right|_{\lambda=0} = i[X(w), \psi(f)].$$

  $X(w) = :\psi^+\psi:(w)$ Wick product. (Quantized field density)

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Goals

- Do the same as above for perturbation W that is non-local in time (as far as possible – might be hard because no Hamiltonian formulation available)
- Example: $(Wf)(x) = \int dy\, w(x,y) f(y)$
- Motivations for that: (1) study "**noncommutative potential scattering**"

$$Wf = w \star f.$$

- (2) **QFT on noncommutative deformations of Minkowski space**. What replaces the "commutative assignment"

$$\mathscr{C}_0^\infty \ni f \longmapsto \psi(f) \in \mathfrak{F} \quad ?$$

- Plan here: Use Bogoliubov's formula to define quantum fields on $\mathbb{R}^n_\theta$.
- Compare to other approaches to QFT on NC spaces
- For commutative time, much of this has been done in [Borris, Verch 2011]

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Precise setup for $D_\lambda = D + \lambda\, W$

Consider

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- D a (pre-)normally hyperbolic operator on $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}^N)$,
- $\lambda \in \mathbb{C}$ a coupling constant,
- W a $\mathscr{C}_0^\infty$-kernel operator:

$$(Wf)(x) = \int dy\, w(x,y) f(y)$$

with $w \in \mathscr{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{N\times N})$.

Then

- Compact support: there exists compact $K \subset \mathbb{R}^n$ such that $W\mathscr{C}^\infty \subset \mathscr{C}_0^\infty(K)$, and $Wf = 0$ for all f with $\operatorname{supp} f \cap K = \emptyset$,
- Smoothing.

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Coupling D and W

- The dynamics of $D + \lambda W$ can be dominated by D or W.
- For large λ, hyperbolic character of D can break down.

Example: Compactly supported solutions

Let

$$(Wf)(x) = \int dy \, (w_1(y), f(y)) \cdot (Dw_2)(x)$$

with $w_1, w_2 \in \mathscr{C}_0^\infty$. Then $f = w_2$ is a solution of D_λ for $\lambda = -\langle w_1, w_2\rangle^{-1}$:

$$D_\lambda w_2 = Dw_2 - \langle w_1, w_2\rangle^{-1}\langle w_1, w_2\rangle Dw_2 = 0\,.$$

- No unique fundamental solutions exist in this case.
- Ambiguities for quantization.
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Formal solution of $D_\lambda f = 0$

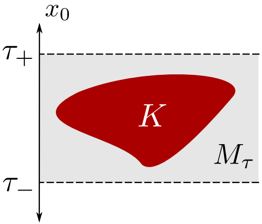
Expanding a solution f_λ of D_λ in a power series in λ, and solving order by order suggests the formula

$$f_\lambda = \sum_{k=0}^{\infty} (-\lambda R^{\pm} W)^k R h \,.$$

Need to control convergence of this series (again, “small λ”).

Fundamental solutions

- For convergence, work first on time slice M_τ.
- $R_\tau^\pm$: Advanced/retarded fundamental solutions of D on M_τ.



Lemma

$R_\tau^\pm W$ and $WR_\tau^\pm$ extend from $\mathscr{C}_0^\infty(M_\tau)$ to bounded operators on $\mathscr{L}^2(M_\tau)$, with image in $\mathscr{C}^\infty(M_\tau)$.

$$N_{\tau,\lambda}^\pm := \sum_{k=0}^\infty (-\lambda R_\tau^\pm W)^k$$

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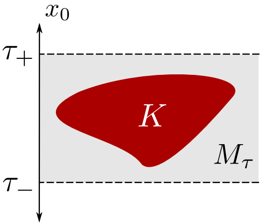

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Fundamental solutions on a time slice

Let $R^{\pm}_{\tau,\lambda} := N^{\pm}_{\tau,\lambda} R^{\pm}_{\tau}$ and $|\lambda|$ small.

Theorem

For any $f \in \mathscr{C}_0^\infty(M_\tau)$,

- $D_{\tau,\lambda} R^{\pm}_{\tau,\lambda} f = f = R^{\pm}_{\tau,\lambda} D_{\tau,\lambda} f.$

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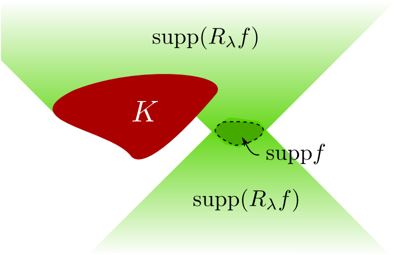

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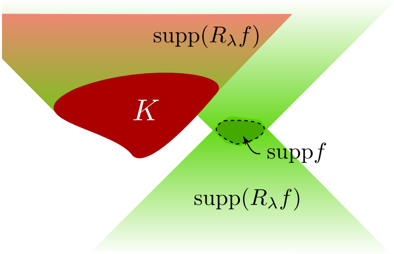

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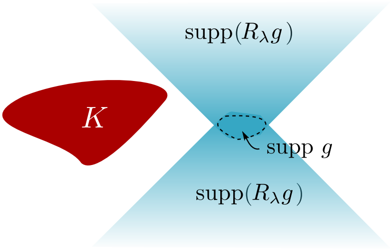
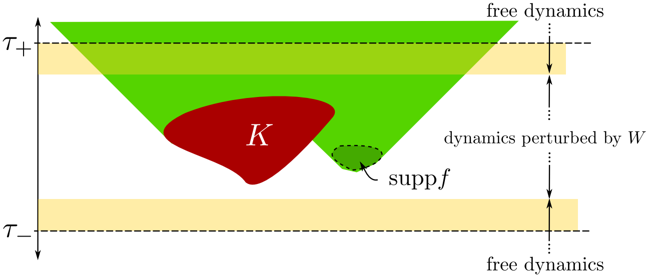
- $\text{supp}(R^{\pm}_{\tau,\lambda} f) \subset \mathcal{J}^{\pm}_{\tau}(\text{supp}\, f) \cup \mathcal{J}^{\pm}_{\tau}(K)$.
- $\text{supp}(R^{\pm}_{\tau,\lambda} f - R^{\pm}_{\tau} f) \subset \mathcal{J}^{\pm}_{\tau}(K)$.
- If $\mathcal{J}^{\pm}_{\tau}(\text{supp}\, f) \cap K = \emptyset$, then $R^{\pm}_{\tau,\lambda} f = R^{\pm}_{\tau} f$



Global fundamental solutions

To extend $R^{\pm}_{\tau,\lambda}$ to all of $\mathbb{R}^n$, "glue" them at the boundary of M_τ.



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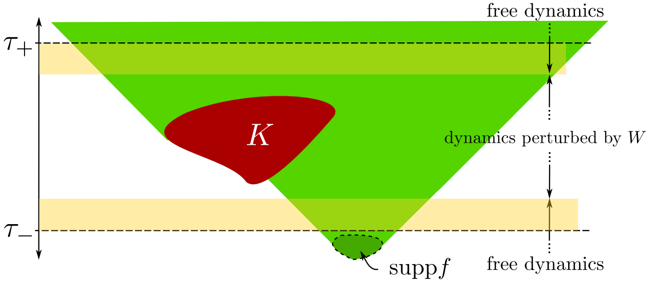

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- These fundamental solutions are unique.

Define the "causal propagator" and solution space

$$R_\lambda := R_\lambda^- - R_\lambda^+ ,$$
$$\mathrm{Sol}_\lambda := \{ f_\lambda \in \mathscr{C}^\infty \,:\, D_\lambda f_\lambda = 0, \quad \operatorname{supp} f_\lambda \text{ spacelike compact } \}$$

- Sol_λ carries a well-defined non-degenerate sesquilinear form

$$\rho_\lambda : \mathrm{Sol}_\lambda \times \mathrm{Sol}_\lambda \to \mathbb{C} , \qquad (R_\lambda f, R_\lambda g) \mapsto \langle f, R_\lambda g \rangle$$

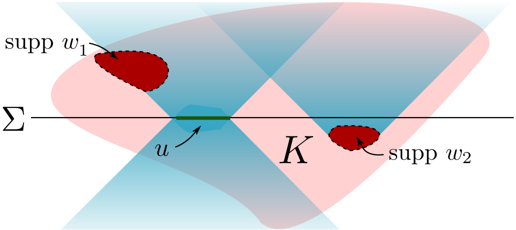
  with

$$\overline{\rho_\lambda(R_\lambda f, R_\lambda g)} = -\rho_\lambda(R_\lambda g, R_\lambda f) .$$

Ill-posed Cauchy Problem

- Even for small λ, the Cauchy problem is ill-posed in general.
- Take for example

$$(D_\lambda f)(x) = (Df)(x) + \lambda \int dy \, (w_1(y), f(y)) \cdot w_2(x)$$



- No solution to the Cauchy problem with Cauchy data u exists.

Well-posed Scattering Problem

- Each solution $f_\lambda \in \mathrm{Sol}_\lambda$ determines two solutions $f_0^\pm \in \mathrm{Sol}_0$ (future/past asymptotics).
- The Møller operators

$$\Omega_{\lambda,\pm} : \mathrm{Sol}_\lambda \to \mathrm{Sol}_0 , \qquad \Omega_{\lambda,\pm} f_\lambda := f_0^\pm$$

  are well-defined linear bijections.
- Define scattering operator

$$S_\lambda := \Omega_{\lambda,+}(\Omega_{\lambda,-})^{-1} : \mathrm{Sol}_0 \to \mathrm{Sol}_0$$

The scattering operator

Theorem

- S_λ is a linear bijection preserving the sesquilinear form ρ_0.
- Explicitly, S_λ is given by

$$S_\lambda = 1 + RW \sum_{k=0}^{\infty} \lambda^{k+1} (-R^+ W)^k \,.$$

- $\lambda \longmapsto S_\lambda f_0$ is analytic in neighborhood of $\lambda = 0$. In particular,

$$\frac{d(S_\lambda f_0)}{d\lambda}\bigg|_{\lambda=0} = RWf_0 \,.$$

Quantization

The solution space $\mathrm{Sol}_\lambda, \rho_\lambda$ can be quantized.

- For $D = D^*$, $W = W^*$, $\lambda \in \mathbb{R}$, the real solutions

$$(\mathrm{Sol}_{\mathbb{R},\lambda}, \rho_\lambda)$$

  form a symplectic space $\to$ CCR quantization.

- For $D =$ Dirac operator and some assumptions on W, the solution space is a pre-Hilbert space $\to$ CAR quantization.

Have C^*-algebras $\mathfrak{F}_0$, $\mathfrak{F}_\lambda$ and isomorphisms

$$\alpha_{\pm,\lambda} : \mathfrak{F}_\lambda \to \mathfrak{F}_0 \text{ induced by Møller operators}$$
$$s_{\pm,\lambda} : \mathfrak{F}_0 \to \mathfrak{F}_0 \text{ induced by scattering operator}$$

But: Local structure of $\mathfrak{F}_\lambda$ and $\mathfrak{F}_0$ quite different! (Different QFTs)

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Perturbations by star product multipliers

Take perturbation of Rieffel product form [Rieffel 92]

$$Wf = w \star f, \qquad w \in \mathscr{C}_0^\infty(K),$$
$$w \star f = \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dy\, e^{2\pi i(p,y)}\, (w \circ \tau_{\theta p}) \cdot (f \circ \tau_y).$$

- τ: action of $\mathbb{R}^n$ on $\mathbb{R}^n$ (τ^* smooth, polynomially bounded)
- θ: real antisymmetric invertible $(n \times n)$-matrix

$w \star f$ is an oscillatory integral taking values in $\mathscr{C}^\infty$ (or $\mathscr{S}$, ...), and $\star$ is a continuous, associative, non-commutative product [GL,Waldmann 2011] [Bieliavsky, Gayral 2011].

$$Wf = \lim_{\varepsilon \to 0} W_\varepsilon f : x \mapsto \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dy\, e^{2\pi i(p,y)}\, \chi(\varepsilon p, \varepsilon y)\; w(\tau_{\theta p}(x)) \cdot f(\tau_y(x))$$

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$$w \star f = \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dy\, e^{2\pi i (p,y)}\, (w \circ \tau_{\theta p}) \cdot (f \circ \tau_y).$$

- τ: action of $\mathbb{R}^n$ on $\mathbb{R}^n$ (τ^* smooth, polynomially bounded)
- θ: real antisymmetric invertible $(n \times n)$-matrix

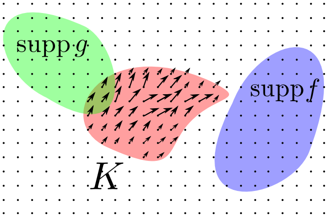
$w \star f$ is an oscillatory integral taking values in $\mathscr{C}^\infty$ (or $\mathscr{S}$, ...), and $\star$ is a continuous, associative, non-commutative product [GL,Waldmann 2011] [Bieliavsky, Gayral 2011].

$$Wf = \lim_{\varepsilon \to 0} W_\varepsilon f : x \mapsto \int_{\mathbb{R}^n} dp \int_{\mathbb{R}^n} dy\, e^{2\pi i (p,y)}\, \chi(\varepsilon p, \varepsilon y)\; w(\tau_{\theta p}(x)) \cdot f(\tau_y(x))$$

with $\chi \in \mathscr{C}_0^\infty$, $\chi(0) = 1$

Perturbations by star product multipliers

- Example 1: $\tau_y(x) = x - y$. Then $\star =$ Moyal product. (have spectral triple structure in this case [Gayral, Gracia-Bondia, Iochum, Schucker, Varilly 2003])
- Example 2: τ such that $\tau_y(K) \subset K$ for all y. "locally noncommutative star product" [GL, Waldmann 2011]



Proposition

- For $\varepsilon > 0$, both $W_\varepsilon^{(1)}$ and $W_\varepsilon^{(2)}$ have $\mathscr{C}_0^\infty$-kernels.
- The kernel of $W^{(1)}$ is smooth, but not compactly supported.

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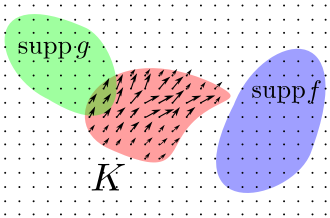

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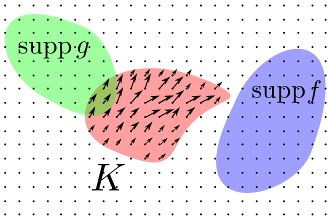

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Bogoliubov's formula for star product multipliers

- For $\varepsilon > 0$, have scattering operators $S_{\varepsilon,\lambda}$ and corresponding automorphisms $s_{\varepsilon,\lambda}$.
- Still have the derivation

$$\lim_{\varepsilon\to 0} \frac{ds_{\varepsilon,\lambda}}{d\lambda}\bigg|_{\lambda=0}$$

on $\mathfrak{F}_0$. Find (for $D = -i\gamma^\mu\partial_\mu$ and Moyal product)

$$\lim_{\varepsilon\to 0} \frac{ds_{\varepsilon,\lambda}(\psi(f))}{d\lambda}\bigg|_{\lambda=0} = i\,[\,:\psi^+\psi:(w) \overset{\star}{,} \psi(f)]$$

with commutator built with Rieffel product.

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Outlook

- Structure of many approaches to NC QFT reemerges from Bogoliubov's formula.
- Also expect formulas of the type

$$\lim_{\varepsilon\to 0} \frac{ds_{\varepsilon,\lambda}(\psi(f))}{d\lambda}\bigg|_{\lambda=0} = i\,[X(w)\,,\psi(f)]$$

  with normal commutator, but deformed field operators $X(w)$.
- Seems to be connected to "warped convolutions" [Grosse, GL 2007], [Buchholz, Summers 2008], [Buchholz, GL, Summers 2011]
  → Module version of Rieffel's deformation procedure
- Also applicable to more general noncommutative structures, such as locally noncommutative products. Remains to be worked out.

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