Dynamics with Histories
(Marseille FFP14)

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Motivations

Find the relation between *symplectic formalism* (time dynamics) and *multisymplectic formalism* (Weyl, de Donder) for field theories.

Find the relation between *multisymplectic* formalism and the *Crnkovic-Witten symplectic* form (on shell).

Requires to work in infinite-dimensional space of sections (histories):
(diffeology inspired) (similarities with Deligne and Freed work).

A generalized synthesis of multisymplectic geometry, the covariant phase space approaches, the canonical approach and the geometry of the space of solutions.

It remains entirely covariant.
Histories furnish the raw material from which reality is constructed (Sorkin)
An *history* is a possible (kinematical) evolution (motion) of the system (becomes a particular solution if it obeys dynamical equations)

**General definition**: an history is an $r$-form on an *evolution domain* $\mathcal{D}$; with values in *configuration space* $Q$ (degrees of freedom of the system);
An *history component* $c$ is a scalar $r$-form on $\mathcal{D}$ ($r$-history);
It belongs to $\Omega^r \mathcal{D} = \text{Sect}[\wedge^r \mathcal{D}] \subset \Omega \mathcal{D}$. 
Histories: examples

In non relativistic (nr) dynamics, $D =$ the time line;
the nr particle, $c : t \rightarrow c(t) = q(t),\quad Q = I \mathbb{R}^3$.

In field theories $D =$ Minkowski spacetime $M$;

- For scalar field, history $c : x \rightarrow x(x) = \varphi(x) =$ scalar function $=$ zero-form $=$ section of the bundle $\bigwedge^0 D = C^0 D$.

- Electromagnetism: history $=$ the Maxwell [scalar] one-form $c = A =$ a section of the bundle $Q = \bigwedge^1 D$.

- First order general relativity: history components are scalar one-forms on the bare manifold $D = M$: the cotetrads one-forms $e^I$ and the spin connection one-forms $\omega^{IJ} =$ sections of $\bigwedge^1 D$. 

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Histories
Velocity-histories

First jet bundle $J^1Q \to Q \to \mathcal{D}$.

**Velocity-history** = Lift of the history $c$ to its first jet extension = the pair $j_1(c) = C = (c, dc)$: a $r$-form and a $(r + 1)$-form.

($d$ is the exterior derivative in $\mathcal{D}$)

(C is nothing but a more explicit way to express $c$).

**ex.** : time dynamics: $dc = \dot{c} \, dt$ and $C$ is usually described as $(c, \dot{c})$.

**ex.** : scalar field $dc = c,_{\mu} \, dx^\mu$, (usually written $\varphi,_{\mu} = \varphi_{\mu}$)

**ex.** : em: $A = c = c_\mu \, dx^\mu$; $dc = A_{\mu\nu} \, dx^{\nu\mu}$ is a 2-form

$C$ is a section of the *configuration-velocity bundle* $\mathbf{V} \to \mathcal{D}$

(subbundle of $J^1Q$)

space of velocity-histories $\mathcal{V} = Sections(\mathbf{V})$.

Lagrangian [historical]dynamics requires differential calculus on $\mathcal{V}$. 
Action $A$: a scalar functional of histories: assigns a real number to each an history

$$A: \begin{array}{c} c \xrightarrow{\text{canonical}} C \rightarrow A[c] = \int_{D} L(C), \end{array}$$

(A solution is an history which makes the action stationary.)

Lagrangian functional

$$L: \begin{array}{c} C = (c, dc) \rightarrow L(c, dc) \in \Omega^{n}D, \end{array}$$

takes the forms $c$ and $dc$ and returns the $n$-form $L(C)$, to be integrated on $D$:

\[\text{def: generalized-functional } \overset{\text{def}}{=} \text{ map } \Omega D \rightarrow \Omega D.\]

Define differential calculus for such functionals / 1
Define \textit{differential calculus for generalized functionals}. \( c \) and \( dc \) play the role of coordinates; (for... \( c(x), dc(x) \))

Define \textbf{exterior derivatives} \( D \) in \( V \) (not \( d \) in \( D \)):

(call \( Dc \) a \([1;r]\)-form; \( dDc = Ddc \) a \([1;(r+1)]\)-form),

Define derivatives:

\[
D\mathcal{L} = Dc \frac{\partial \mathcal{L}}{\partial c} + D(dc) \frac{\partial \mathcal{L}}{\partial dc}
\]

(1)

(explicit formulas in paper)
examples

\[ F = F_\alpha \ldots d^{\alpha} \ldots \quad c = c_\mu d^\mu \ldots \]

\[
\frac{\partial F}{\partial c} = \frac{\partial F_\alpha}{\partial c_\mu \ldots} (e_\mu \ldots \lrcorner d^\alpha) \quad (2)
\]

dual components: \( P = P^\mu \text{Vol}_\mu \). Then,

\[
\frac{\partial F}{\partial P} = \varepsilon^{\mu \ldots \nu} \quad \frac{\partial F_\alpha}{\partial P^\nu} (e_\nu \lrcorner d^\alpha). \quad (3)
\]
Consider an arbitrary \textit{infinitesimal variation} of history as [the result of] action of a \textit{vector-field} \( \delta \) in \( \mathcal{V} \):

\[
\delta c = \langle Dc, \delta \rangle, \quad \delta (dc) = \langle D(dc), \delta \rangle
\]

(usual action of 1-form on vector-field)

\[
\Rightarrow \quad \delta(L) = \langle DL, \delta \rangle = \delta c \frac{\partial L}{\partial c} + \delta (dc) \frac{\partial L}{\partial dc}
\]

\( \rightarrow \) well posed variational problem
Define \( \frac{\delta^{\text{EL}} L}{\delta c} \equiv \frac{\partial L}{\partial c} - (-1)^{|c|} \frac{d}{\partial (dc)} \frac{\partial L}{\partial (dc)} \), \hspace{1cm} (4) \\

Then, \( D L = D c \left( \frac{\delta^{\text{EL}} L}{\delta c} \right) - d \Theta \). \\

with \( \Theta \equiv -D c \ P = D c \left( \frac{\partial L}{\partial (dc)} \right) \).

generalized [1;(n-1)] \text{symplectic potential} (see below) : (cf. \( \theta = p \ dq \))
Universal motion equations

\[ D\mathcal{L} = Dc \left( \frac{\delta^{EL}\mathcal{L}}{\delta c} \right) - d\Theta \Rightarrow \frac{\partial \mathcal{L}}{\partial c} - (-1)^r d \left( \frac{\partial \mathcal{L}}{\partial dc} \right) = 0; \]

→ usual equations for any dynamics:
\[ td : \quad dc \star dc = (\dot{c} \ dt) \ \dot{c} \rightarrow \Box c = \ddot{q} = 0.\ldots \]
\[ sc.\text{fied} : \quad dc \star dc = (\phi_\mu \ \ dx^\mu) \star d\phi = \phi_\mu \ \phi^\mu \ \text{Vol} \ \rightarrow \ \Box c = \Box \phi = 0.\ldots \]
\[ em : \quad dc \star dc = dA \star dA = A_\mu^\nu \ A^{\mu\nu} \ \text{Vol} \ \rightarrow \ \Box c = \Box A = 0.\ldots \]
...see general relativity below
Historical momentum

Define *Historical momentum* \( P \overset{\text{def}}{=} \frac{\partial L}{\partial d c}(C) : \) a generalized functional \( C \rightarrow P(C) \): value is an \((n-r-1)\)-form.

Natural expression in dual components \( P \overset{\text{def}}{=} \frac{\partial L}{\partial d c} = P^\mu \ldots \text{Vol}^\mu \ldots ; \quad (\text{Vol}^\mu \ldots \overset{\text{def}}{=} e^\mu \ldots \int \text{Vol}.) \)

**time dynamics** : \( P = \dot{p} \)

**Fields** : \( P^\mu = " \text{polymomenta} " \ldots \) (used in polysymplectic formalism)

...
Generalized symplectic structure

\[
D\mathcal{L} = Dc \frac{\delta^{EL}\mathcal{L}}{\delta c} - d\Theta
\]

\(\Theta \overset{\text{def}}{=} - Dc P \Rightarrow \) (by derivation) \(\omega \overset{\text{def}}{=} D\Theta = DP \wedge Dc:\)  

*generalized symplectic form* \([2:(n-1)]\)

*historical version* of the symplectic structure on \(TM\) (or of the pre-symplectic structure of the evolution space.)
True symplectic structure on shell

The generalized symplectic form is conserved on shell: Motion equations imply $d\bar{\omega} = 0$:
(covariant conservation of the symplectic current on shell). Integration of $\bar{\omega}$ along a $(n-1)$-dimensional submanifold of $\mathcal{D}$ gives a scalar valued two-form $\omega$ such that $D\omega = 0$
Conservation (above) implies $d\omega = 0$:
$\Rightarrow \omega$ does not depend (on shell) on the choice of the hypersurface (assumed time-like for FTs).
$\rightarrow \omega$ is a genuine [scalar-valued] canonical symplectic form on the space of solutions. (cf. Crnkovic-Witten)
Electromagnetism in Minkowski spacetime

history is a one-form \( c = A \).
Lagrangian functional \( \mathcal{L} : A \rightarrow \frac{1}{2} \, dA \, (\star dA) \) (Hodge duality in Minkowski spacetime).
It results \( \frac{\partial \mathcal{L}}{\partial (dA)} = \star dA \) and the Euler-Lagrange equation

\[ d(\star dA) = 0 \iff \Box A = 0. \]
First order general relativity

First order Lagrangian functional

\[ \mathcal{L} = \epsilon_{IJKL} \ e^I \ e^J \ (d \ \omega^{KL} + (\omega \omega)^{KL}). \]  

(wedge product). Dynamical variables: the \textit{cotetrad one-forms} \( e^I \) and the \textit{Lorentz-connection one-forms} \( \omega^{KL} \).

\[ \frac{\partial \mathcal{L}}{\partial e^I} = 2 \ \epsilon_{IJKL} \ e^J \ (d \ \omega^{KL} + (\omega \omega)^{KL}) \quad \text{and} \quad P_i \overset{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial d e^I} = 0; \]  

(7)

\[ \frac{\partial \mathcal{L}}{\partial \omega^{KL}} = 2 \ \epsilon_{IJNL} \ e^I \ e^J \ \omega^N_K = 2 \ \epsilon_{IJKL} \ e^I \ \omega^J_M \ e^M \]  

(8)

\[ \Pi_{KL} \overset{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial d \omega^{KL}} = \epsilon_{IJKL} \ e^I \ e^J. \]  

(9)
First order general relativity/2

The EL equations for each variable:

\[
\frac{\partial L}{\partial e^I} = 2 \epsilon_{IJKL} e^J (d \omega^{KL} + (\omega \omega)^{KL}) = 0
\]

which means zero Ricci curvature; and

\[
\frac{\partial L}{\partial \omega^{KL}} + d \frac{\partial L}{\partial d\omega^{KL}} = \epsilon_{IJKL} e^I [\omega^J_M e^M + de^J] = 0
\]

which means zero torsion.
Hamiltonian formalism: sketch

- take $c$ and $P$ as new variables (instead of $c$, $dc$) ("coordinate change" through a generalized Legendre map);
- **Historical Hamiltonian** ($n$-form-valued generalized functional)

$$\mathcal{H} = \Lambda^i \Gamma_i + \Pi \text{Vol} + P \, dc - \mathcal{L}$$

(constraints and Lagrange mult. also generalized functionals)

- universal Hamilton equations **COVARIANT**.

\[
\begin{align*}
dc &= \frac{\partial H}{\partial P}; \\
dP &= -\frac{\partial H}{\partial c}.
\end{align*}
\] (10)
universal Hamilton equations give usual Hamilton equations in time dynamics. A Coordinate expansion gives the Hamilton De DonderWeyl equations for field theories. concretely gives immediately the (usual) solutions in electromagnetism, first order general relativity... There is a **COVARIANT** generalized symplectic form $DP \wedge Dc$. conserved on shell $\rightarrow$ symplectic form on shell! links other approaches...
Conclusion

- Transfer of the usual formalisms to the infinite dimensional space of histories (sections rather than fiber bundles)
- Dynamics takes an *universal formulation*, always *covariant* in particular, no space+time splitting for field theories.
- Lagrangian-like and Hamiltonian-like formulations
- Universal motion equations in both cases
- Generalized symplectic form $\rightarrow$ Poisson like bracket for observables
- Work in progress: quantization, quantum gravity
THANK YOU