Dynamics with Histories (Marseille FFP14)

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Motivations

Find the relation between *symplectic formalism* (time dynamics) and *multisymplectic formalism* (Weyl, de Donder) for field theories.

Find the relation between *multisymplectic* formalism and the *Crnkovic-Witten symplectic* form (on shell).

Requires to work in infinite-dimensional space of sections (histories) :

(diffeology inspired) (similarities with Deligne and Freed work).

A generalized synthesis of multisymplectic geometry, the covariant phase space approaches, the canonical approach and the geometry of the space of solutions.

It remains entirely covariant.

Histories

Histories furnish the raw material from which reality is constructed (Sorkin)

An *history* is a possible (kinematical) evolution (motion) of the system (becomes a particular solution if it obeys dynamical equations)

General definition : **an history is an r-form**
on an evolution domain $\mathcal{D}$;
with values in *configuration space* Q (degrees of freedom of the system);

An *history component* c is a scalar r-form on $\mathcal{D}$ (r-history) ;
It belongs to $\Omega^r\mathcal{D} = Sect[\bigwedge^r \mathcal{D}] \subset \Omega\mathcal{D}$.

Histories: examples

In non relativistic **(nr) dynamics**, $\mathcal{D}=$ the time line;

the **nr particle**, $c: \ t \to c(t) = q(t), \quad Q = \mathbb{R}^3$.

In **field theories** $\mathcal{D}$=Minkowski spacetime M ;

- For **scalar field**, history $c: \ x \to x(x) = \varphi(x) =$ scalar function $=$ zero-form $=$ section of the bundle $\bigwedge^0 \mathcal{D} = \mathcal{C}^0\mathcal{D}$.
- *Electromagnetism*: history $=$ the Maxwell [scalar] one-form $c = A =$ a section of the bundle $\mathbf{Q} = \bigwedge^1 \mathcal{D}$.
- *First order general relativity*: history components are scalar one-forms on the bare manifold $\mathcal{D} = M$: the cotetrads one-forms e^I and the spin connection one-forms $\omega^{IJ} =$ sections of $\bigwedge^1 \mathcal{D}$

Velocity-histories

First jet bundle $J^1\mathbf{Q} \to \mathbf{Q} \to \mathcal{D}$.
Velocity-history = Lift of the history c to its first jet extension = the pair $j_1(c) = C = (c, \mathrm{d}c)$: a r-form and a $(r+1)$-form.
(d is the exterior derivative in $\mathcal{D}$)
(C is nothing but a more explicit way to express c).
ex. : time dynamics : $\mathrm{d}c = \dot{c}\ \mathrm{d}t$ and C is usually described as $(c, \dot{c})$.
ex. : scalar field $\mathrm{d}c = c_{,\mu}\ \mathrm{d}x^\mu$, (usually written $\varphi_{,\mu} = \varphi_\mu$)
ex. : em: $A = c = c_\mu\ \mathrm{d}x^\mu$; $\mathrm{d}c = A_{\mu\nu}\ \mathrm{d}x^{\nu\mu}$ is a 2-form
C is a section of the *configuration-velocity bundle* $\mathbf{V} \to \mathcal{D}$
(subbundle of $J^1\mathbf{Q}$)
space of velocity-histories $\mathcal{V} = \mathit{Sections}(\mathbf{V})$.
Lagrangian [historical]dynamics requires differential calculus on $\mathcal{V}$.

action and Lagrangian

Action $\mathcal{A}$: a scalar functional of histories:
assigns a real number to each an history

$$\mathcal{A}: \; c \overset{canonical}{\rightarrow} C \rightarrow \mathcal{A}[c] = \int_{\mathcal{D}} \mathcal{L}(C),$$

(A solution is an history which makes the action stationary.)
Lagrangian functional

$$\mathcal{L}: \; C = (c, \mathrm{d}c) \rightarrow \mathcal{L}(c, \mathrm{d}c) \in \Omega^n \mathcal{D}$$

takes the forms c and $\mathrm{d}c$ and returns the n-form $\mathcal{L}(C)$,
to be integrated on $\mathcal{D}$:

def: generalized-functional $\overset{def}{=}$ map $\Omega\mathcal{D} \rightarrow \Omega\mathcal{D}$.

Define **differential calculus for such functionals / 1**

differential calculus for functionals

Define *differential calculus for generalized functionals* .
c and $\mathrm{d}c$ play the role of coordinates ; (for... c(x), dc(x))
Define **exterior derivatives** D in $\mathcal{V}$ (not d in $\mathcal{D}$):
(call $\mathrm{D}c$ a [1;r]-form ; $\mathrm{dD}c = \mathrm{Dd}c$ a [1;(r+1)]-form),
Define derivatives:

$$\boxed{\mathrm{D}\mathcal{L} = \mathrm{D}c \ \frac{\partial \mathcal{L}}{\partial c} + \mathrm{D}(\mathrm{d}c) \ \frac{\partial \mathcal{L}}{\partial \mathrm{d}c}} \tag{1}$$

(explicit formulas in paper)

examples

$$F = F_{\alpha\dots}\mathrm{d}^{\alpha\dots} \qquad c = c_{\underline{\mu}}\ \mathrm{d}^{\mu\dots}$$

$$\frac{\partial F}{\partial c} = \frac{\partial F_{\alpha\dots}}{\partial c_{\underline{\mu}\dots}}\ (e_{\underline{\mu}}\dots \lrcorner\, \mathrm{d}^{\alpha}) \tag{2}$$

dual components: $P = P^{\underline{\mu}}\ \mathrm{Vol}_{\underline{\mu}}$. Then,

$$\frac{\partial F}{\partial P} = \epsilon^{\underline{\mu}\dots\underline{\nu}\dots}\ \ \frac{\partial F_{\underline{\alpha}}}{\partial P^{\underline{\mu}}}\ (e_{\underline{\nu}} \lrcorner\, \mathrm{d}^{\underline{\alpha}}). \tag{3}$$

Variations = vector-field

Consider an arbitrary *infinitesimal variation* of history as [the result of] action of a *vector-field* δ in $\mathcal{V}$:

$$\delta c = \langle \mathrm{D}c, \delta \rangle, \qquad \delta(\mathrm{d}c) = \langle \mathrm{D}(\mathrm{d}c), \delta \rangle$$

(usual action of 1-form on vector-field)

$$\Rightarrow \ \delta(\mathcal{L}) = \langle \mathrm{D}\mathcal{L}, \delta \rangle = \delta c \ \frac{\partial \mathcal{L}}{\partial c} + \delta(\mathrm{d}c) \ \frac{\partial \mathcal{L}}{\partial \mathrm{d}c}$$

$\rightarrow$ well posed variational problem

Euler–Lagrange derivative

$$\textit{Define} \quad \frac{\delta^{EL}\mathcal{L}}{\delta c} \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial c} - (-1)^{|c|} \, \mathrm{d} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}c)}, \tag{4}$$

$$\boxed{\textit{Then,} \qquad \mathrm{D}\mathcal{L} = \mathrm{D}c \, \left(\frac{\delta^{EL}\mathcal{L}}{\delta c}\right) - \mathrm{d}\Theta.} \tag{5}$$

$$\textit{with } \Theta \stackrel{def}{=} -\mathrm{D}c \; P = \mathrm{D}c \, \frac{\partial \mathcal{L}}{\partial(\mathrm{d}c)}.$$

generalized [1;(n-1)] *symplectic potential* (see below) : (cf. $\theta = p \; \mathrm{d}q$)

Universal motion equations

$$\mathrm{D}\mathcal{L} = \mathrm{D}c\ (\frac{\delta^{EL}\mathcal{L}}{\delta c}) - \mathrm{d}\Theta \Rightarrow \frac{\partial\mathcal{L}}{\partial c} - (-1)^r \mathrm{d}\left(\frac{\partial\mathcal{L}}{\partial \mathrm{d}c}\right) = 0;$$

$\rightarrow$ usual equations for any dynamics:
td : $\mathrm{d}c \star\mathrm{d}c = (\dot{c}\ \mathrm{d}t)\ \dot{c} \rightarrow \Box c = \ddot{q} = 0 \ldots.$
sc.fied : $\mathrm{d}c \star\mathrm{d}c = (\phi_\mu\ \mathrm{d}x^\mu) \star\mathrm{d}\varphi = \phi_\mu\ \phi^\mu$ Vol $\rightarrow$
$\Box c = \Box\varphi = 0 \ldots.$
em : $\mathrm{d}c \star\mathrm{d}c = \mathrm{d}A\star\mathrm{d}A = A_{\mu\nu}\ A^{\mu\nu}$ Vol $\rightarrow \Box c = \Box A = 0 \ldots.$
...see general relativity below

Historical momentum

Define *Historical momentum* $P \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial \mathrm{d}c}(C)$:
a generalized functional $C \to P(C)$: value is an (n-r-1)-form.
Natural expression in dual components $P \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial \mathrm{d}c} = P^{\mu\cdots} \mathrm{Vol}_{\mu\ldots}$;

$$(\mathrm{Vol}_{\mu\ldots} \stackrel{def}{=} e_{\mu}\ldots \lrcorner \mathrm{Vol}.)$$

time dynamics : $P = \dot{p}$
Fields : $P^{\mu} =$ " polymomenta "... (used in polysymplectic formalism)

...

Generalized symplectic structure

$$\mathrm{D}\mathcal{L} = \mathrm{D}c \; \left(\frac{\delta^{EL}\mathcal{L}}{\delta c}\right) - \mathrm{d}\Theta$$

$\Theta \overset{def}{=} -\mathrm{D}c\; P \Rightarrow$(by derivation) $\quad \overline{\omega} \overset{def}{=} \mathrm{D}\Theta = \mathrm{D}P \wedge \mathrm{D}c$:
generalized symplectic form [2:(n-1)]
historical version of the symplectic structure on $\mathrm{T}M$ (or of the pre-sympletic structure of the evolution space.)

True symplectic structure on shell

The generalized symplectic form is conserved on shell:
Motion equations imply $\mathrm{d}\overline{\omega} = 0$:
(covariant conservation of the symplectic current on shell).
Integration of $\overline{\omega}$ along a $(n-1)$-dimensional submanifold of $\mathcal{D}$ gives a scalar valued two-form ω such that $\mathrm{D}\omega = 0$
Conservation (above) implies $\mathrm{d}\omega = 0$:
$\Rightarrow \omega$ does not depend (on shell) on the choice of the hypersurface (assumed time-like for FTs).
$\rightarrow \omega$ is a genuine [scalar-valued] canonical symplectic form on the space of solutions. (cf. Crnkovic-Witten)

Electromagnetism in Minkowski spacetime

history is a one-form $c = A$.
Lagrangian functional $\mathcal{L}: \ A \rightarrow \frac{1}{2} \, \mathrm{d}A \, (\star \mathrm{d}A)$ (Hodge duality in Minkowski spacetime).
It results $\frac{\partial \mathcal{L}}{\partial (\mathrm{d}A)} = \star \mathrm{d}A$ and the Euler-Lagrange equation

$$\mathrm{d}(\star \mathrm{d}A) = 0 \Leftrightarrow \Box A = 0.$$

First order general relativity

First order Lagrangian functional

$$\mathcal{L} = \epsilon_{IJKL}\ e^I\ e^J\ (\mathrm{d}\ \omega^{KL} + (\omega\omega)^{KL}). \tag{6}$$

(wedge product). Dynamical variables: the *cotetrad one-forms* e^I and the *Lorentz-connection one-forms* ω^{KL}.

$$\frac{\partial \mathcal{L}}{\partial e^I} = 2\ \epsilon_{IJKL}\ e^J\ (\mathrm{d}\ \omega^{KL} + (\omega\omega)^{KL}) \text{ and } P_I \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial \mathrm{d} e^I} = 0; \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \omega^{KL}} = 2\ \epsilon_{IJNL}\ e^I\ e^J\ \omega^N{}_K = 2\ \epsilon_{IJKL}\ e^I\ \omega^J{}_M\ e^M \tag{8}$$

$$\Pi_{KL} \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial \mathrm{d}\omega^{KL}} = \epsilon_{IJKL}\ e^I\ e^J. \tag{9}$$

First order general relativity/2

The EL equations for each variable :

$$\frac{\partial \mathcal{L}}{\partial e^I} = 2\ \epsilon_{IJKL}\ e^J\ (\mathrm{d}\ \omega^{KL} + (\omega\omega)^{KL}) = 0$$

which means zero Ricci curvature; and

$$\frac{\partial \mathcal{L}}{\partial \omega^{KL}} + \mathrm{d}\frac{\partial \mathcal{L}}{\partial \mathrm{d}\omega^{KL}} = \epsilon_{IJKL}\ e^I\ [\omega^J{}_M\ e^M + \quad de^J] = 0$$

which means zero torsion.

Hamiltonian formalism: sketch

- take c and P as new variables (instead of c, dc)
("coordinate change" through a generalized Legendre map);
- *Historical Hamiltonian* (n-form-valued generalized functional)

$$\mathcal{H} = \Lambda^i \, \Gamma_i + \Pi \, \mathrm{Vol} + P \, \mathrm{d}c - \mathcal{L}$$

(constraints and Lagrange mult. also generalized functionals)
- universal Hamilton equations COVARIANT.

$$\boxed{\mathrm{d}c = \frac{\partial \mathcal{H}}{\partial P}; \quad \mathrm{d}P = -\frac{\partial \mathcal{H}}{\partial c}.} \tag{10}$$

Hamiltonian formalism: sketch / 2

universal Hamilton equations give usual Hamilton equations in time dynamics. A Coordinate expansion gives the Hamilton De DonderWeyl equations for field theories.
concretely gives immediately the (usual) solutions in electromagnetism, first order general relativity...
There is a COVARIANT generalized symplectic form $\mathrm{D}P \wedge \mathrm{D}c$.
conserved on shell $\rightarrow$ symplectic form on shell !
links other approaches...

Conclusion

- Transfer of the usual formalisms to the infinite dimensional space of histories (sections rather than fiber bundles)
- Dynamics takes an universal formulation, always covariant in particular, no space+time splitting for field theories.
- Lagrangian-like and Hamiltonian-like formulations
- Universal motion equations in both cases
- generalized symplectic form $\rightarrow$ Poisson like bracket for observables
- work in progress: quantization, quantum gravity

THANK YOU