Physical predictions from lattice QCD

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How to stay clean in the brown muck

Purpose of lattice QCD

- QCD fundamental objects: quarks and gluons
- QCD observed objects: protons, neutrons (π, K, ...)

Huge discrepancy: not even the same particles observed as in the Lagrangean

Perturbation theory has no chance

Need to solve low energy QCD to:
- Compute hadronic and nuclear properties
  “people who love QCD”
  - Masses, decay widths, scattering lengths, thermodynamic properties, ...
- Compute hadronic background
  “people who hate QCD”
  - Non-leptonic weak MEs, quark masses, g-2, ...
LATTICE DISCRETIZATION

- UV cutoff: space-time lattice
- Hypercubic, spacing $a$
- Momentum cutoff $p_\mu < 2\pi/a$
- IR cutoff on finite lattice

- Anti-commuting quark fields $\psi(x)$ live on the sites
- Gluon fields $U_\mu(x) = e^{ig\int_{x}^{x+e_\mu} dz_\mu A_\mu(z)} \in SU(3)$ live on links

Essential:
- QCD perturbative on cutoff scale $1/a \gg \Lambda_{QCD}$ (asymptotic freedom)
- Perform Euclidean path integral stochastically
Lattice

Lattice QCD = QCD when

- Cutoff removed (continuum limit)
- Infinite volume limit taken
- At physical hadron masses (Especially $\pi$)
  - Numerically challenging to reach light quark masses

Statistical error from stochastic estimate of the path integral
Landscape $M_\pi$ vs. $a$

- ETMC '09 (2)
- ETMC '10 (2+1+1)
- MILC '10
- MILC '12
- QCDSF '10 (2)
- QCDSF-UKQCD '10
- BMWc '10
- BMWc'08
- PACS-CS '09
- RBC/UKQCD '10
- JLQCD/TWQCD '09
- HSC '08
- BGR '10
- CLS '10 (2)
Landscape $L$ vs. $M_\pi$
### Landscape $M_K$ vs. $M_\pi$

![Graph showing the landscape of $M_K$ vs. $M_\pi$.](image-url)

- **ETMC '10 (2+1+1)**
- **MILC '10**
- **QCDSF-UKQCD '10**
- **BMWc'10**
- **PACS-CS '09**
- **JLQCD/TWQCD '09**
- **RBC-UKQCD '10**
- **HSC '08**

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**Physical predictions from lattice QCD**
Lattice setup

Skeleton of a lattice calculation

- Compute target observable
- Extrapolate to physical point
- Renormalize if necessary
We have done our homework

Ground state mass extraction

With operators that couple to the ground state (e.g. to the $\pi^+$)

$$A_\mu(t) = \sum_{\vec{x}} \left( \bar{\psi}^d \gamma_\mu \gamma_5 \psi^u \right)(\vec{x}, t)$$

one can obtain asymptotically the ground state mass

$$C(t) = \langle A_0^\dagger(t) A_0(0) \rangle \xrightarrow{t \to \infty} \frac{|\langle \pi | A_0 | 0 \rangle|^2}{2M_\pi} e^{-M_\pi t}$$

$$\ln \frac{C(t)}{C(t+1)} \xrightarrow{t \to \infty} M_{\pi}^{\text{eff}}$$
We have done our homework

Effective masses and correlated fits
We have done our homework

Chiral fit

![Graph showing physical predictions from lattice QCD](image)
We have done our homework

Finite volume effects

The easy part: Virtual pion finite V effects

- Hadrons see mirror charges
- Exponential in lightest particle (pion) mass
- Leading effects \( \frac{M_X(L) - M_X}{M_X} = cM^{1/2}_\pi L^{-3/2} e^{M_\pi L} \) (Colangelo et. al., 2005)

More severe (if present):

- FV correction in resonances
- QED: \( 1/L \) terms from photons (Davoudi, Savage 2014; BMWc 2014)
We have done our homework

Systematic uncertainties

Method:

- Large number of analyses including “all reasonable” choices
- Construct (weighted) distribution of results
  - Median of this distribution ➔ final result
  - Central 68% ➔ systematic error
- Statistical error from bootstrap of the medians
We have done our homework

The light hadron spectrum

(BMWc, 2008)
We have done our homework

Excited states

Extracting excited states is much tougher:

- Extraction of energy levels is harder:
  Die out at large $t$ \( \Rightarrow \) need to use small $t$ correlators
- Once extracted, relation to $V \rightarrow \infty$ is nontrivial:
  Disentangle resonances and scattering states at finite volume

\[\text{Finite volume energy levels} \quad \text{Spectral density}\]
Relevance of fine structure

Is the fine structure relevant?

- Proton, neutron: 3 quarks
- Proton: uud
- Neutron: udd

- $m_u < m_d$: $M_p < M_n$
- $m_u = m_d$: $M_p > M_n$

Proton decays

$M_p + M_{e^-} \gtrsim M_n$

No hydrogen
Relevance of fine structure

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- $m_u < m_d : M_p < M_n$
- $m_u = m_d : M_p > M_n$
- Proton decays
- $M_p + M_e^- \geq M_n$
- No hydrogen
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Proton, Neutron
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Proton

Neutron
Anthropic puzzle? The light up quark

1\textsuperscript{st} generation: \(m_u < m_d\)

2\textsuperscript{nd} generation: \(m_c > m_s\)

3\textsuperscript{rd} generation: \(m_t > m_b\)
Sources of isospin splitting

Two sources of isospin breaking:

- **QCD:** \( \sim \frac{m_d - m_u}{\Lambda_{QCD}} \sim 1\% \)
- **QED:** \( \sim \alpha (Q_u - Q_d)^2 \sim 1\% \)

On the lattice:

- Include nondegenerate light quarks \( m_u \neq m_d \)
- Include QED
Challenges of QED simulations

- Effective theory only (UV completion unclear)
- $\pi^+$, $\rho$, etc. no more gauge invariant
- QED (additive) mass renormalization
- Power law FV effects (soft photons)

EM field of a point charge cannot be made periodic & continuous  
Remove $\vec{p} = 0$ modes in fixed gauge (Hayakawa, Uno, 2008)
Finite volume subtraction

- Universal to $O(1/L^2)$
- Divergent $T$ dependence for $p = 0$ mode subtraction
- No $T$ dependence for $\vec{p} = 0$ mode subtraction

\[ \delta m = q^2 \alpha \left( \frac{\kappa}{2mL} \left( 1 + \frac{2}{mL} - \frac{3\pi}{(mL)^3} \right) \right) \]

(BMWc, 2014)
Isospin splitting

\[ \Delta M \text{[MeV]} \]

- \( \Delta \Sigma \)
- \( \Delta \Xi \)
- \( \Delta N \)
- \( \Delta \Xi_{cc} \)
- \( \Delta D \)
- \( \Delta_{CG} \)

Experiment
QCD+QED
Prediction

(BMWc 2014)
Disentangling contributions

Problem:
- Disentangle QCD and QED contributions
  - Not unique, $O(\alpha^2)$ ambiguities
- Flavor singlet (e.g. $\pi^0$) difficult (disconnected diagrams)

Method:
- Use baryonic splitting $\Sigma^+ - \Sigma^-$ purely QCD
  - Only physical particles
  - Exactly correct for pointlike particle
  - Corrections below the statistical error
Nucleon splitting QCD and QED parts

(BMWc 2014)
Light quark masses

Goal:
- Compute light quark masses ab initio
- Alternative view:
  Translate fundamental parameters ($M_\pi$, $M_K$) into perturbatively useful quantities

Method:
- Go to the physical point
- Read off input quark masses and renormalize

Challenge:
- Minimize and control all systematics
  - 2+1 dynamical fermion flavors
  - Physical quark masses
  - Continuum extrapolation
  - Infinite volume
  - Nonperturbative renormalization
Quark masses

Renormalization

- Quark masses logarithmically divergent \((a \to 0)\) \rightarrow renormalization
- Usually \(\overline{\text{MS}}\) scheme: only perturbatively defined

\(\overline{\text{RI-MOM}}\) scheme

- Matrix elements of off-shell quarks in fixed gauge

\[
\begin{align*}
\frac{Z_S^{\overline{\text{RI-MOM}}}(\mu^2)}{Z_0^0} & = 1 - \frac{\beta_0}{2} \ln \frac{\mu^2}{\Lambda_{\overline{\text{MS}}}^2} \\
\end{align*}
\]

Renormalization condition: at \(p^2 = \mu^2\) tree level matrix element
Continuum extrapolation

The graph shows the relationship between the quark mass $m_s(4\text{GeV})$ in MeV and the lattice spacing $a$ in fm. The data points are plotted at different lattice spacings $a=0.06\text{ fm}$, $a=0.08\text{ fm}$, $a=0.10\text{ fm}$, and $a=0.12\text{ fm}$. The graph indicates a linear trend as the lattice spacing decreases, suggesting a continuum extrapolation. The $x$-axis represents the lattice spacing $a$ in fm, and the $y$-axis represents the quark mass $m_s(4\text{GeV})$ in MeV.
## Quark masses

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Publication status</th>
<th>Chiral extrapolation</th>
<th>Continuum extrapolation</th>
<th>Finite volume</th>
<th>Renormalization</th>
<th>$m_{ud}$</th>
<th>$m_s$</th>
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<td>C</td>
<td></td>
<td></td>
<td>a</td>
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<td>A</td>
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<td>★★</td>
<td>b</td>
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<tr>
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<td>O</td>
<td>★★</td>
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<td>94.2(1.4)(3.2)(4.7)</td>
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<tr>
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<td>★★</td>
<td>★★</td>
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<td>86.7(2.3)</td>
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<td>O</td>
<td>★★</td>
<td>a</td>
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<td>3.44(12)(22)</td>
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<td>★</td>
<td>★★</td>
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<td></td>
<td></td>
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<td>107.3(4.4)(9.7)(4.9)</td>
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<td>★</td>
<td>★★</td>
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</table>
# Quark masses

<table>
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<th>$m_u$</th>
<th>$m_d$</th>
<th>$m_u/m_d$</th>
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<td>★ □ □ ★ a</td>
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<td>-</td>
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<td>-</td>
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<td>-</td>
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<td>□ □ ★ ☆</td>
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<td>3.02(27)(19)</td>
<td>5.49(20)(34)</td>
<td>0.550(31)</td>
</tr>
</tbody>
</table>

*(FLAG group, 2013)*
Pseudoscalar decay constant

With the axial vector current

$$A_{\mu}(t) = \sum_{\vec{x}} \left( \bar{\Psi}^d \gamma_\mu \gamma_5 \Psi^u \right)(\vec{x}, t)$$

one obtains

$$\langle A_0^\dagger(t)A_0(0) \rangle \xrightarrow{t \to \infty} \frac{|\langle \pi | A_0 | 0 \rangle|^2}{2M_\pi} e^{-M_\pi t} = \frac{M_\pi^2 F_{\pi}^{\text{bare}}^2}{2M_\pi} e^{-M_\pi t}$$
Chiral behavior

\[
\frac{\chi^2}{\# \text{dof}} = 1.52, \ #\text{dof} = 17, \ \text{p-value} = 0.08
\]

Exp value (NOT INCLUDED)
$F_{\pi} = 92.9(9)(2)$

(BMWc 2014)
Matrix element of an **effective weak operator** (e.g. \( \langle K^\dagger | O | K \rangle \)):

\[
\langle J_{0}^{L\dagger}(t_{+})O(0)J_{0}^{L}(t_{-}) \rangle \xrightarrow{t_{\pm} \rightarrow \pm \infty} \frac{\langle K | J_{0}^{L} | 0 \rangle |^{2}}{(2M_{K})^{2}} \langle K^{\dagger} | O | K \rangle e^{-M_{K}(t_{+}-t_{-})}
\]

where

\[
J_{0}^{L} = [\bar{s}d]_{V-A} = \bar{s} \gamma_{0} (1 - \gamma_{5}) d
\]

\[
O_{\Delta S=2} = [\bar{s}d]_{V-A}[\bar{s}d]_{V-A}
\]

Norm from:

\[
\langle J_{0}^{L\dagger}(t)J_{0}^{L}(0) \rangle \xrightarrow{t \rightarrow \infty} \frac{\langle K | J_{0}^{L} | 0 \rangle |^{2}}{2M_{K}} e^{-M_{K}t}
\]
Unphysical operator mixing

$\chi$SB induces mixing with 4 unphysical operators

Mixing terms chirally enhanced

Small even below physical $m_\pi$

Good chirality of our action
Matrix elements

Physical point

![Graph showing physical predictions from lattice QCD](image)
Matrix elements

Continuum extrapolation

\[ B_K^{RI}(3.5 \text{ GeV}) \]

\[ \alpha_s a [\text{fm}] \]

Physical predictions from lattice QCD

Christian Hoelbling (Wuppertal)
Matrix elements

Errors

\[ B_{K}^{RI} (3.5 \text{ GeV}) \]

**statistical**

**systematic**

\[ 0.51 \ 0.52 \ 0.53 \ 0.54 \ 0.55 \]

\[ \text{statistical systematic} \]

36/39 Christian Hoelbling (Wuppertal)
Physical predictions from lattice QCD
Matrix elements

UT prediction (CKM Fitter, 2012)
ETMC (2009, TM)
RBC-UKQCD (2010, DW)
Aubin et al. (2010, DW/MILC)
Laiho, Van de Water (2011, STAG/MILC)
SWME (2011, HYP-STAG/MILC)
SWME (2014, HYP-STAG/MILC)
BMW-c (2011, 2 HEX-CIW)
Progress since 2008

Physical predictions from lattice QCD

(BMWc 2014)
THANK YOU
BACKUP
Action details

Goal:

- Optimize physics results per CPU time
- Conceptually clean formulation

Method:

- Dynamical 2 + 1 flavor, Wilson fermions at physical $M_\pi$
- 3-5 lattice spacings $0.053 \text{ fm} < a < 0.125 \text{ fm}$
- Tree level $O(a^2)$ improved gauge action (Lüscher, Weisz, 1985)
- Tree level $O(a)$ improved fermion action (Sheikholeslami, Wohlert, 1985)
  - Why not go beyond tree level?
    - Keeping it simple (parameter fine tuning)
    - No real improvement, UV mode suppression took care of this
  - This is a crucial advantage of our approach
- Discretization effects of $O(\alpha_s a, a^2)$
  - We include both $O(\alpha_s a)$ and $O(a^2)$ into systematic error
Algorithm stability

![Graph showing forces and their stability]
No exceptional configs

Inverse iteration count \(\frac{1000}{N_{cg}}\)

- \(\beta = 3.31, M_\pi \approx 135\) MeV
- \(\beta = 3.5, M_\pi \approx 130\) MeV
- \(\beta = 3.61, M_\pi \approx 120\) MeV
- \(\beta = 3.7, M_\pi \approx 180\) MeV
- \(\beta = 3.8, M_\pi \approx 220\) MeV
Topological sector sampling

Topological charge $\beta=3.8$, $m_{ud}=-0.02$, $m_s=0$

worst case
Autocorrelation time (finest lattice, small mass)

\[ \tau_{\text{int}} = 27.3(7.4) \]

(MATLAB code from Wolff, 2004-7)
Locality properties

- Locality in position space:
  \[ |D(x, y)| < \text{const } e^{-\lambda |x-y|} \text{ with } \lambda = O(a^{-1}) \text{ for all couplings.} \]
  Our case: \( D(x, y) = 0 \) as soon as \( |x - y| > 1 \)
  (despite smearing)

- Locality of gauge field coupling:
  \[ |\delta D(x, y)/\delta A(z)| < \text{const } e^{-\lambda |(x+y)/2 - z|} \text{ with } \lambda = O(a^{-1}) \text{ for all couplings.} \]
  Our case: \( \delta D(x, x)/\delta A(z) < \text{const } e^{-\lambda |x - z|} \text{ with } \lambda \approx 2.2a^{-1} \text{ for } 2 \leq |x - z| \leq 6 \)
Gauge field coupling locality

6-stout case:

\[
\frac{\partial D(x,y)}{\partial U_\mu(x+z)}
\]

![Graph showing the gauge field coupling locality for different lattice spacings.](image)

- \(a \approx 0.125 \text{ fm}\)
- \(a \approx 0.085 \text{ fm}\)
- \(a \approx 0.065 \text{ fm}\)
Chiral interpolation

- Simultaneous fit to NLO $SU(2)\chi$PT (Gasser, Leutwyler, 1984)
- Consistent for $M_\pi \lesssim 400$ MeV

We use 2 safe chiral interpolation ranges: $M_\pi < 340, 380$ MeV

We use $SU(2)\chi$PT and Taylor interpolation forms
High lying resonances

(Bulava, et al., 2010)

- Qualitative understanding of experimental spectrum
- No extrapolation to physical point, continuum
Including isospin breaking on the lattice

\[ S_{QCD+QED} = S_{QCD}^{iso} + \frac{1}{2}(m_u - m_d) \int (\bar{u}u - \bar{d}d) + ie \int A_\mu j_\mu \]

with \( j_\mu = \bar{q}Q\gamma_\mu q \)

Method 1: operator insertion (RM123 ’12-’13)

\[ \langle O \rangle = \langle O \rangle_{QCD}^{iso} - \frac{1}{2}(m_u - m_d) \langle O \int (\bar{u}u - \bar{d}d) \rangle_{QCD}^{iso} + \frac{1}{2} e^2 \langle O \int_{xy} j_\mu(x)D_{\mu\nu}(x - y)j_\nu(y) \rangle_{QCD}^{iso} + \ldots \]

✓ Use existing ensembles
✓ Going beyond \( O(m_u - m_d) \) or \( O(\alpha) \) difficult, but rarely necessary
✗ Complicated observables and renormalization
✗ Disconnected diagrams
Lattice isospin breaking

Method 2: direct calculation

✔ Complete solution, observables “as usual”

✗ Only partially implemented yet

- \( m_u \neq m_d \) valence only (MILC ’09, BMWc ’10-, Blum et al ’10, RBC/UKQCD ’12, ...)
- QED valence only (Eichten ’97, Blum et al ’07-, BMWc ’10-, MILC ’10-)
- \( m_u \neq m_d \) (PACS-CS ’12) and QED (Blum et al ’12) reweighting

✔ Valence approximation reasonable

- Errors \( O(\alpha\alpha_s), O((m_u - m_d)^2) \)

In this talk: QCD and QED isospin breaking, valence only

- Compact QED, Coulomb gauge \( \rightarrow \) linear
Parameterization

Problem:
- Parameterize QCD and QED splitting

Method:
- Use $\Delta M^2 = M_{uu}^2 - M_{dd}^2$ to parameterize QCD splitting
- Use $\alpha_{QED}$ to parameterize QED splitting

$$\Delta M_X = \Delta M^2 C_X + \alpha_{QED} D_X$$

$$C_X = c_X^0 + c_X^1 \hat{M}_\pi^2 + c_X^2 \hat{M}_K^2 + c_X^3 f(a)$$

$$D_X = d_X^0 + d_X^1 \hat{M}_\pi^2 + d_X^2 \hat{M}_K^2 + d_X^3 a + d_X^4 \frac{1}{L}$$

- WIP: use physical (hadronic) definition
Finite volume corrections

\[ \Delta M^2_K \] vs. \( \frac{1}{L} \) [MeV]

- Blue: fit
- Red: \( a = 0.11 \) fm
- Green: \( a = 0.09 \) fm
- Blue: \( a = 0.07 \) fm
- Red: \( a = 0.06 \) fm
- Green: \( a = 0.05 \) fm

\( \chi = 0.11 \) fm
\( \chi = 0.09 \) fm
\( \chi = 0.07 \) fm
\( \chi = 0.06 \) fm
\( \chi = 0.05 \) fm
Interpolation in $\alpha$

![Graph showing interpolation in $\alpha$ with different values of $\alpha$ and corresponding $\Delta M^2_K$ values in [MeV^2].]
Chiral behavior

![Graph showing physical predictions from lattice QCD]

- $a = 0.11$ fm
- $a = 0.09$ fm
- $a = 0.07$ fm
- $a = 0.06$ fm
- $a = 0.05$ fm

$\Delta M_K^2$ [MeV$^2$] vs $M_{\pi^+}^2$ [MeV$^2$]
Strategy outline

Goal:
- Compute light quark masses ab initio

Relevance:
- Fundamental SM parameters
- Stability of matter depends on their values
- Not obtainable perturbatively

Challenge:
- Minimize and control all systematics
  - 2+1 dynamical fermion flavors
  - Physical quark masses
  - Continuum extrapolation
  - Nonperturbative renormalization
  - Infinite volume
Renormalization strategy

Goal:
- Full nonperturbative renormalization
- Optional accurate conversion to perturbative scheme

Method:
- We use RI-MOM scheme (Martinelli et al., 1993)
  - $O(a)$ correction (Maillart, Niedermayer, 2008)
- Compute $m_q$ at low scale $\mu \ll 2\pi/a \sim 11 − 24$ GeV
  - $\mu = 2.1$ GeV
  - $\mu = 1.3$ GeV
- Do continuum non-perturbative running to high scale $\mu' \gg \Lambda_{\text{QCD}}$
- Further conversion in 4-loop PT
Desired scale in RI-MOM scheme
Reaching the perturbative regime

![Graph showing the ratio of RI nonpert to RI,4-loop normalized to Z, as a function of \( \mu^2 \). The graph includes a horizontal line at \( Z = 1 \) and a vertical line at \( \mu = 4 \) GeV.]
Optional conversion to $\overline{\text{MS}}$

![Graph showing the conversion of ZS(2-loop)/ZS(1-loop), ZS(3-loop)/ZS(2-loop), ZS(4-loop)/ZS(3-loop), and ZS(4-loop/ana)/ZS(4-loop) to $\overline{\text{MS}}$. The graph plots the ratios of these quantities against the energy scale in GeV.]
Quark mass definitions

- **Lagrangian mass** $m^{\text{bare}}$
  - $m^{\text{ren}} = \frac{1}{Z_S} (m^{\text{bare}} - m_{\text{crit}}^{\text{bare}})$

- **$m^{\text{PCAC}}$ from** \( \frac{\langle \partial_0 A_0 P \rangle}{\langle P(t)P(0) \rangle} \)
  - $m^{\text{ren}} = \frac{Z_A}{Z_P} m^{\text{PCAC}}$

Better use

- **$d = m^{\text{bare}} - m_{ud}^{\text{bare}}$**
  - $d^{\text{ren}} = \frac{1}{Z_S} d$

and reconstruct

- **$m_s^{\text{ren}} = \frac{1}{Z_S} \left( \frac{r d}{r-1} \right)$**
- **$m_{ud}^{\text{ren}} = \frac{1}{Z_S} \left( \frac{d}{r-1} \right)$**

- No additive mass renormalization and ambiguity in $m_{\text{crit}}$
- Only $Z_S$ multiplicative renormalization (no pion poles)
- Works with $O(a)$ improvement (we use this)
Tiny finite volume effects

- FV effects tiny
- Dedicated FV runs
- Perfect agreement with FV $\chi$PT (Colangelo et al. 2005)

\[ M_{\pi} L = 3 \quad M_{\pi} L = 4 \]

ignored in final analysis

\[ aM_{\pi} \]

\[ L/a \]

16 24 32
Light quark masses

\[ m_{ud}^{RI}(4\text{GeV}) \ [\text{MeV}] \]

\[ \alpha a \ [\text{fm}] \]

- \( a = 0.06 \text{ fm} \)
- \( a = 0.08 \text{ fm} \)
- \( a = 0.10 \text{ fm} \)
- \( a = 0.12 \text{ fm} \)

\[ \frac{63}{39} \text{ Christian Hoelbling (Wuppertal)} \]

Physical predictions from lattice QCD
Individual $m_u$ and $m_d$

- **Goal:**
  - Compute $m_u$ and $m_d$ separately

- **Method:**
  - Need QED and isospin breaking effects in principle
  - Alternative: use dispersive input $-Q$ from $\eta \rightarrow \pi\pi\pi$
  
  $$Q^2 = \frac{1}{2} \left( \frac{m_s}{m_{ud}} \right)^2 \frac{m_d - m_u}{m_{ud}}$$

  - Transform precise $m_s/m_{ud}$ into $(m_d - m_u)/m_{ud}$
  - We use the conservative $Q = 22.3(8)$ (Leutwyler, 2009)
Systematic error treatment

- **Goal:**
  - Reliably estimate total systematic error

- **Method:**
  - 288 full analyses (2000 bootstrap on each)
    - 2 plateaux regions
    - 2 continuum forms: $O(\alpha_s a), O(a^2)$
    - 3 chiral forms: $2 \times SU(2)$, Taylor
    - 2 chiral ranges: $M_\pi < 340, 380$ MeV
    - 3 renormalization matching procedures
    - 2 NP continuum running forms
    - 2 scale setting procedures
  - All analyses weighted by fit quality
    - Mean gives final result
    - Stdev gives systematic error
  - Statistical error from 2000 bootstrap samples
## Final result

<table>
<thead>
<tr>
<th></th>
<th>RI @ 4 GeV</th>
<th>RGI</th>
<th>MS @ 2 GeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_s$</td>
<td>96.4(1.1)(1.5)</td>
<td>127.3(1.5)(1.9)</td>
<td>95.5(1.1)(1.5)</td>
</tr>
<tr>
<td>$m_{ud}$</td>
<td>3.503(48)(49)</td>
<td>4.624(63)(64)</td>
<td>3.469(47)(48)</td>
</tr>
<tr>
<td>$m_u$</td>
<td>2.17(04)(10)</td>
<td>2.86(05)(13)</td>
<td>2.15(03)(10)</td>
</tr>
<tr>
<td>$m_d$</td>
<td>4.84(07)(12)</td>
<td>6.39(09)(15)</td>
<td>4.79(07)(12)</td>
</tr>
</tbody>
</table>

### Additional consistency checks:

- Use $m^{\text{PCAC}}$ only, no ratio-difference method: $\checkmark$
  - compatible, slightly larger error
- Unweighted final result and systematic error: $\checkmark$
  - negligible impact
- Additional Continuum, chiral and FV terms: $\checkmark$
  - all compatible with 0
Comparison

\[ m_{ud} \]

- PACS-CS 09
- HPQCD 09
- MILC 09A
- MILC 09
- PACS-CS 08
- RBC/UKQCD 08
- CP-PACS 07
- MILC 07
- HPQCD 05
- MILC 04

\[ m_s \]

- PACS-CS 09
- HPQCD 09
- MILC 09A
- MILC 09
- PACS-CS 08
- RBC/UKQCD 08
- CP-PACS 07
- MILC 07
- HPQCD 05
- MILC 04

FLAG estimate

- Dominguez 09
- Chetyrkin 06
- Jamin 06
- Narison 06
- Vainshtein 78

This work

- MILC 09
- MILC 09A
- HPQCD 09
- PACS-CS 09
- PDG 09

67/39 Christian Hoelbling (Wuppertal) Physical predictions from lattice QCD
## Standard model neutral K mixing

### Goal:
- Check SM CP violation in neutral K system

### Method:
- Compute effective weak matrix element
- Relate kaon CP violation to CKM phase

### Challenge:
- Minimize and control all systematics
  - 2+1 dynamical fermion flavors
  - Physical quark masses
  - Mixing of unphysical operators
  - Continuum
  - Infinite volume
Signal

![Signal Graph]

\[ B_{K}^{\text{bare}(\tau)} \]

Physical predictions from lattice QCD
Running

\[
R_{R_i}\nu (\mu, 3.5 \text{ GeV})/R_{B_k}\nu (\mu, 3.5 \text{ GeV})
\]

\[
\alpha_s\text{-scaling}
\]

\[
\alpha_s^2\text{-scaling}
\]

\[
\mu^2 [\text{GeV}^2]
\]
NLO fit, x-expansion

\[ \chi^2 \frac{\text{# dof}}{\text{dof}} = 1.52, \text{#dof} = 17, \text{p-value} = 0.08 \]

Exp value (NOT INCLUDED)
NNLO fit, x-expansion

\[ \frac{\chi^2}{\text{dof}} = 1.16, \ #\text{dof} = 48, \ p\text{-value} = 0.21 \]

Exp value (NOT INCLUDED)
NLO fit, $\xi$-expansion

$\chi^2$ # dof = 1.54, #dof = 17, p-value = 0.07

Physical predictions from lattice QCD
NNLO fit, $\xi$-expansion

$\chi^2 / \text{#dof} = 1.19$, #dof = 48, p-value = 0.18
Extracting $F$

![Graph showing data points and error bars for $F$ and $\Delta F$ as functions of $M_{\pi}^{max}$]
Extracting $F$
Extracting $F$

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xlabel={$M_{\pi}^{\text{min}}$ [MeV]},
    ylabel={p-value},
    xmin=100, xmax=350,
    ymin=1e-06, ymax=10,
    xtick={100,150,200,250,300},
    ytick={0.0001,0.001,0.01,1,10},
    legend pos=north east
]
\addplot[mark=triangle,red] coordinates {
(150,0.0001)
(200,0.0001)
(300,0.0001)
};
\addlegendentry{NLO $x$}
\addplot[mark=circle,blue] coordinates {
(150,0.001)
(200,0.001)
(300,0.001)
};
\addlegendentry{NLO $\xi$}
\end{axis}
\end{tikzpicture}
\end{center}
Extracting $F$

![Graph showing the extraction of $F$ and $\Delta F$ as functions of $M_{\pi}^{\text{min}}$. The graph includes data points for NLO and NLO $\xi$.](image-url)