A fluid of diffusing particles and its cosmological behaviour

Zbigniew Haba
Institute of Theoretical Physics, University of Wroclaw

Content

I. RHS of Einstein equations
II. Relativistic diffusion on RHS
III.Homogeneous metric
IV. Non-homogeneous metric
V. Temperature fluctuations

Classical equations

The Einstein equations ($g^{\mu\nu}$ is the metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$)

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}, \tag{1}$$

with

$$(T^{\mu\nu})_{;\mu} = 0. \tag{2}$$

RHS of Einstein equations

- phase space distribution of particles
- fields
- fluids

Phase space distribution

The phase-space distribution satisfies Liouville equation (for geodesic motion, here $\Gamma^{\mu}_{\nu\rho}$ are Christoffel symbols)

$$(p^{\mu}\partial^{x}_{\mu} - \Gamma^{k}_{\mu\nu}p^{\mu}p^{\nu}\partial_{k})\Phi(x,p) = 0. \tag{3}$$

The formula for the energy-momentum tensor is

$$\tilde{T}^{\mu\nu} = \sqrt{g}\int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{p^0}p^{\mu}p^{\nu}\Phi, \tag{4}$$

$\tilde{T}$ is conserved
g is the determinant of the metric and p^0 is determined from the mass-shell condition $p_{\mu}p^{\mu} = m^2$ (m is the particle's mass, we set $c = 1$). Greek indices run from 0 to 3, Latin indices denoting spatial components have the range from 1 to 3, the covariant derivative is over the space-time, derivatives over the momenta $\frac{\partial}{\partial p^k}$ are denoted ∂_k and ∂^x denotes a derivative over a space-time coordinate x.

Fluids

Assuming we have a phase space distribution we can define

$$v^\mu = \langle p^\mu \rangle \tag{5}$$

Then,

$$\langle p^\mu p^\nu \rangle = \langle 1 \rangle v^\mu v^\nu + \langle (p^\mu - v^\mu)(p^\nu - v^\nu) \rangle \tag{6}$$

Let u^μ be a normalized v^μ ,i.e.

$$g_{\mu\nu} u^\mu u^\nu = 1 \tag{7}$$

Then, the identity for $\langle p^\mu p^\nu \rangle$ can be expressed as

$$T^{\mu\nu} = E u^\mu u^\nu - \pi (g^{\mu\nu} - u^\mu u^\nu) + \pi^{\mu\nu} \tag{8}$$

Fields

If we have the action W then

$$T_{\mu\nu} = \frac{\delta W}{\delta g^{\mu\nu}(x)} \tag{9}$$

For the scalar field

$$W = \int dx \sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{10}$$

Classical scalar fields are applied to generate inflation

What if the energy-momentum is not conserved?

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = T^{\mu\nu} = T_D^{\mu\nu} + \tilde{T}^{\mu\nu}, \tag{11}$$

where T_D is the energy-momentum of a certain (dark) matter and $\tilde{T}$ is the energy-momentum of the system of diffusing particles. From the lhs it follows that

$$(T_D^{\mu\nu})_{;\mu} = -(\tilde{T}^{\mu\nu})_{;\mu}. \tag{12}$$

Knowing the rhs of we can determine the lhs up to a constant. We represent T_D by a time-dependent cosmological term Λ. A dynamical relation of the cosmological term to the matter density seems to be unavoidable for an explanation of the coincidence problem.

Why diffusion?

- **Diffusion equilibrates to a temperature (equilibrium) state washing out initial conditions**

The diffusion on the mass-shell

$$g_{\mu\nu}p^{\mu}p^{\nu} = m^2 \tag{13}$$

The diffusion is generated by the Laplace-Beltrami operator on $\mathcal{H}_+$

$$\triangle_H = \frac{1}{\sqrt{G}}\partial_j G^{jk}\sqrt{G}\partial_k \tag{14}$$

where

$$G^{jk} = m^2 g^{jk} + p^j p^k \tag{15}$$

$\partial_j = \frac{\partial}{\partial p^j}$ and $G = \det(G_{jk})$ is the determinant of G_{jk}.

The transport equation for the linear diffusion generated by $\triangle_H$ reads

$$(p^\mu \partial^x_\mu - \Gamma^k_{\mu\nu} p^\mu p^\nu \partial_k)\Omega = \kappa^2 \triangle_H \Omega, \tag{16}$$

where κ^2 is the diffusion constant, $\partial^x_\mu = \frac{\partial}{\partial x^\mu}$ and $x = (t, \mathbf{x})$

Quantum phase space distributions

If the phase space distribution has the Bose-Einstein or Fermi-Dirac equilibrium limit which is a minimum of the relative entropy (related to the free energy) then the diffusion equation must be non-linear. The proper generalization reads

$$(p^{\mu}\partial_{\mu}^{x}-\Gamma_{\mu\nu}^{k}p^{\mu}p^{\nu}\partial_{k})\Omega=\kappa^{2}p_{0}\partial_{j}\Big(G^{jk}p_{0}^{-1}\partial_{k}\Omega+\beta p^{j}\Omega(1+\nu\Omega)\Big), \tag{17}$$

where $\nu=1$ for bosons and $\nu=-1$ for fermions. The classical (Boltzmann) statistics can be described by $\nu=0$.

Solutions of linear and non-linear diffusion equations at finite temperature

We have the time-dependent equilibrium

$$\Omega_E^{PL} = \Big(\exp(\beta a^2(p+\mu)) - \nu\Big)^{-1} \tag{18}$$

where μ is an arbitrary constant (the chemical potential). In the ultrarelativistic limit (a large p) the Planck distribution is the same as the Jüttner distribution.
For the equilibrium solution we obtain standard Friedmann cosmology.
There are other solutions of the diffusion equation whose energy momentum tensor gives different Friedmann equation for the scale factor a

We solve the conservation equation for Λ then with $H = a^{-1}\frac{da}{d\tau}$ the Friedmann equation reads

$$\frac{3}{8\pi G}H^2 \equiv \frac{3}{8\pi G}(a^{-1}\frac{da}{d\tau})^2 = \tilde{T}^{00}(\tau) - \int_{\tau_0}^{\tau} dr a^{-4}\partial_r(a^4\tilde{T}^{00}) + \frac{\Lambda}{8\pi G}(\tau_0)$$
$$= \tilde{T}^{00}(\tau_0) - 4\int_{\tau_0}^{\tau} drH(r)\tilde{T}^{00}(r) + \frac{\Lambda}{8\pi G}(\tau_0) \tag{19}$$

Here, $\tilde{T}^{\mu\nu}$ energy-momentum of diffusing particles and any conserved energy-momentum (e.g. scalar fields),
$T_D^{\mu\nu} = \Lambda g^{\mu\nu}$
and (τ is the cosmic time)
$ds^2 = d\tau^2 - a^2(\tau)d\mathbf{x}^2$

Explicit solution

We can find an explicit power-like solution of the integro-differential equation by a fine tuning of parameters showing that the exponential behaviour is not a necessity even if $\Lambda(\tau_0) > 0$. Let us assume

$$a(\tau) = \nu(\tau - q)^{\gamma} \tag{20}$$

with the initial condition $a(\tau_0) = \nu(\tau_0 - q)^{\gamma}$. Inserting we determine the parameters

$$\gamma = 1, \tag{21}$$

$$\nu = \sigma^{\frac{1}{3}}, \tag{22}$$

$$(\tau_0 - q)^2 = \frac{2\theta}{\nu} \tag{23}$$

Then

$$\Lambda(\tau_0) = \frac{3}{2}(\tau_0 - q)^{-2}. \tag{24}$$

We obtain

$$\Lambda = 8\pi G\tilde{E} = \frac{3}{2}(\tau - q)^{-2} \tag{25}$$

Non-homogeneous metric:Fluctuation spectrum

$$\langle\frac{\delta T}{T}(\mathbf{n})\frac{\delta T}{T}(\mathbf{n}')\rangle = \sum_{l=0}^{\infty}(2l+1)C_l P_l(\mathbf{nn}') \tag{26}$$

n direction in the sky

Experimental result: COBE, WMAP

$$l(l+1)C_l \simeq const \tag{27}$$

ordinary Sachs-Wolfe effect

Einstein-Liouville-Vlasov equations

We decompose

$$g_{\mu\nu} = \overline{h}_{\mu\nu} + h_{\mu\nu} \tag{28}$$

where $\overline{h}_{\mu\nu}$ describes homogenous metric in the conformal time

$$ds^2 = \overline{h}_{\mu\nu} dx^\mu dx^\nu = a^2(dt^2 - d\mathbf{x}^2) \tag{29}$$

and

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2\Big((1+2\phi)dt^2 - (1+2\psi)d\mathbf{x}^2 - \gamma_{ij} dx^i dx^j\Big) \tag{30}$$

For massless particles ($m = 0$) and in the homogeneous metric $h_{\mu\nu} = 0$ the Jüttner distribution

$$\Omega_E = \exp(-a^2\beta|\mathbf{p}|) \tag{31}$$

with

$$\mathbf{p}^2 = \sum_j p^j p^j$$

is the solution of Liouville eq.

Non-homogeneous metric

The solution can be expressed as

$$\Omega = \Omega_E^g + \beta p_0 \Theta \Omega_E^g \tag{32}$$

where

$$\Omega_E^g = \exp(-\beta p_0) \tag{33}$$

and p_0 is determined from $g_{\mu\nu} p^\mu p^\nu = 0$

Θ is the solution of the equation

$$\partial_t \Theta + n^k \partial_k^x \Theta = -2\partial_t \psi - \frac{1}{2} n^j n^k \partial_t \gamma_{jk} \tag{34}$$

where

$$n^k = p^k |\mathbf{p}|^{-1}$$

We have

$$\Omega = \exp(-\frac{p_0}{T + \delta T}) \tag{35}$$

Let

$$\frac{\delta T}{T} = \Theta \tag{36}$$

Then

$$\begin{aligned}\Theta(t, \mathbf{x}) = \Theta_0(\mathbf{x} - \mathbf{n}t) - \textstyle\int_0^t \Big(2\partial_s \psi(s, \mathbf{x} - (t-s)\mathbf{n}) \\ + \tfrac{1}{2}\partial_s \gamma_{jk}(s, \mathbf{x} - (t-s)\mathbf{n}) n^j n^k\Big) ds\end{aligned} \tag{37}$$

with the initial condition $\Theta_0(\mathbf{x})$.

Diffusive space-time temperature variation

We look for solutions in the form

$$\Omega = \Omega_E^g + \beta p_0 \Theta \Omega_E^g + ra^2 \Omega_E^g = (1 + ra^2) \exp(-\frac{p_0}{T + T\Theta}) \quad (38)$$

Inserting this formula in the diffusion equation we obtain equations for the temperature variation Θ and r

$$\partial_t \Theta + n^k \partial_k^x \Theta + \kappa^2 \beta a^2 \Theta = -2\partial_t \psi - \frac{1}{2} n^j n^k \partial_t \gamma_{jk} \quad (39)$$

$$\partial_t r + n^k \partial_k^x r = 3\kappa^2 \Theta \quad (40)$$

where

$$n^k = p^k |\mathbf{p}|^{-1} \quad (41)$$

The solution reads

$$\Theta_t(\mathbf{x}) = \exp(-\beta\kappa^2 \int_0^t a^2(s)ds)\Theta_0(\mathbf{x} - t\mathbf{n})$$
$$- \int_0^t ds \exp(-\beta\kappa^2 \int_s^t a^2(r)dr)(2\partial_s\psi(s, \mathbf{x} - (t-s)\mathbf{n}) \tag{42}$$
$$+\frac{1}{2}\partial_s\gamma_{jk}n^j n^k(s, \mathbf{x} - (t-s)\mathbf{n})$$

Temperature fluctuations

We restrict ourselves to tensor perturbations

$$\begin{aligned}&\langle\Theta(t,\mathbf{n})\Theta(t,\mathbf{n}')\rangle = \\ &= \frac{1}{4}(2\pi)^{-3}\int_0^t ds\int_0^t ds'\int d\mathbf{q}F(s,s',q)\exp(-\beta\kappa^2(\int_s^t+\int_{s'}^t)dra^2(r)) \\ &(2(\mathbf{n}\Delta(\mathbf{q})\mathbf{n}')^2-(\mathbf{n}\Delta(\mathbf{q})\mathbf{n})(\mathbf{n}'\Delta(\mathbf{q})\mathbf{n}'))\exp(-i(t-s)\mathbf{nq}+i(t-s')\mathbf{n}'\mathbf{q})\end{aligned} \tag{43}$$

where

$$\mathbf{n}\Delta(\mathbf{q})\mathbf{n}' = \mathbf{n}\mathbf{n}' - \mathbf{q}^{-2}(\mathbf{q}\mathbf{n})(\mathbf{q}\mathbf{n}') \equiv \Delta(\mathbf{n}\mathbf{n}', \mathbf{e}\mathbf{n}, \mathbf{e}\mathbf{n}') \qquad (44)$$

$$\mathbf{n}\Delta(\mathbf{q})\mathbf{n} = 1 - \mathbf{q}^{-2}(\mathbf{q}\mathbf{n})^2 \equiv \Delta(\mathbf{e}\mathbf{n}) \qquad (45)$$

where we write $\mathbf{q} = q\mathbf{e}$ and

$$F(s, s', q) = \partial_s \partial_{s'} P(s, s', \mathbf{q}) \qquad (46)$$

P is the expectation value of tensor perturbations.

If the power spectrum F is known then there remains to perform the integrals over s and q in order to obtain

$$\begin{aligned}&\langle\Theta(t,\mathbf{n})\Theta(t,\mathbf{n}')\rangle == \sum_{l=0}^{\infty}(2l+1)\tilde{D}_l(t,\mathbf{nn}')P_l(\mathbf{nn}')\\&=\sum_{l=0}^{\infty}(2l+1)C_l(t)P_l(\mathbf{nn}')\end{aligned} \tag{47}$$

where P_l are the Legendre polynomials and $\tilde{D}_lP_l$ still must be expanded in Legendre polynomials if the coefficients C_l are to be independent of the angles. We have

$$\begin{aligned}&\tilde{D}_l == \frac{1}{16\pi^2}\int_0^t ds\int_0^t ds'\int d\mathbf{q}F(s,s',q)\exp(-\beta\kappa^2(\int_s^t+\int_{s'}^t)dra^2(r))\\&\Big(2\Delta(\mathbf{nn}',-i\partial_s,i\partial_{s'})^2-\Delta(-i\partial_s)\Delta(i\partial_{s'})\Big)j_l(q(t-s))j_l(q(t-s'))\end{aligned} \tag{48}$$

Conclusions

1.Diffusion gives a damping factor for temperature variation
2. The damping factor for temperature fluctuations can give a finite result even if perturbation is acting for an infinite time
3. The diffusion can change the l behavior of multipoles