Causal Structure for Noncommutative Geometry

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Noncommutative geometry à la Connes = spectral triples

1. Algebratisation of the Riemannian geometry
2. Testing the concepts - new noncommutative horizons
3. Applications - particle physics, cosmology, ...

Drawbacks of the standard spectral approach

- Relativistic physics is Lorentzian rather than Riemannian
- We loose the causal structure
- Applications - need for a Wick rotation \((t \rightarrow it)\)

Lorentzian spectral triples - a remedy?

1. Algebratisation of the causal structure
2. Testing the concepts - almost-commutative space-times
3. Applications - ... ?
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The axioms of noncommutative geometry

\((\mathcal{A}, \mathcal{H}, \mathcal{D})\) - spectral triple

- \(\mathcal{A}\) - pre-\(C^*\)-algebra (unital)
- \(\mathcal{H}\) - Hilbert space
  \(\exists\) a faithful representation \(\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})\)
- \(\mathcal{D}\) - the Dirac operator - selfadjoint, unbounded
  - \((\mathcal{D} - \lambda)^{-1}\) for any \(\lambda \notin \mathbb{R}\) - compact resolvent
  - \([\mathcal{D}, \pi(a)] \in \mathcal{B}(\mathcal{H})\) for all \(a \in \mathcal{A}\)
- \(\ldots\)
- The spectrum of Lorentzian \(\mathcal{D}\) is way more complicated
The axioms of noncommutative geometry

\((A, \mathcal{H}, D)\) - spectral triple

- \(A\) - pre-\(C^*\)-algebra (unital)
- \(\mathcal{H}\) - Hilbert space
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- \(D\) - the Dirac operator - selfadjoint, unbounded
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  - \([D, \pi(a)] \in B(\mathcal{H})\) for all \(a \in A\)
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Lorentzian spectral triples

\( (\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, D, J) \) - Lorentzian spectral triple

- A Hilbert space \( \mathcal{H} \).
- A non-unital pre-\( C^* \)-algebra \( \mathcal{A} \) with a faithful representation on \( B(\mathcal{H}) \).
- A preferred unitisation \( \tilde{\mathcal{A}} \) of \( \mathcal{A} \) which is a pre-\( C^* \)-algebra with a faithful representation on \( \mathcal{H} \) and such that \( \mathcal{A} \) is an ideal of \( \tilde{\mathcal{A}} \).
- An unbounded operator \( D \) densely defined on \( \mathcal{H} \) such that:
  - \( \forall a \in \tilde{\mathcal{A}} \ [D, a] \) extends to a bounded operator on \( \mathcal{H} \),
  - \( \forall a \in \mathcal{A} \ a(\Delta_J^{-1}) \) is compact, with \( \Delta_J := (\frac{1}{2}(DD^* + D^*D) + 1)^{1/2} \).
- A bounded operator \( J \) on \( \mathcal{H} \) - fundamental symmetry - such that:
  - \( J^2 = 1, J^* = J \),
  - \( [J, a] = 0 \ \forall a \in \tilde{\mathcal{A}} \),
  - \( D^* = -J D \),
  - \( J \) captures the Lorentzian signature of the metric \( [N. Franco, M.E. (2014b)]: J = -N[D, T] \), with \( N \in \tilde{\mathcal{A}}^+, T \in \mathcal{L}(\mathcal{H}) \).
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  - \( \forall a \in \tilde{A} \) \( [D, a] \) extends to a bounded operator on \( H \),
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Globally hyperbolic manifold

A commutative Lorentzian spectral triple on a Lorentzian manifold $M$:

- $\mathcal{A} \subset C_0^\infty(M)$ – smooth functions vanishing at $\infty$.
- $\tilde{\mathcal{A}} \subset C_b^\infty(M)$ – smooth bounded functions with all derivatives bounded.
- $\mathcal{H} = L^2(M, S)$ – space of square integrable spinor sections over $M$.
- Non-degenerate products on $\mathcal{H}$:
  - Indefinite: $(f, g) = \int_M (f_x, g_x)x \sqrt{|g|} d^n x$.
  - Positive definite: $\langle f, g \rangle := (f, J_r g)$.

- Spacelike reflection $r \in \text{Aut}(TM)$, $r^2 = 1$, $g(r \cdot, r \cdot) = g(\cdot, \cdot)$
  $g^r(\cdot, \cdot) := g(\cdot, r \cdot)$ - positive definite metric on $TM = F^- \oplus F^+$
- $\tilde{J}_r$ - fundamental symmetry associated with $r$
  $$\tilde{J}_r c(e_0) \tilde{J}_r = -c(re_0), \quad \tilde{J}_r = ic(e_0) = i\gamma^0$$

- $D = -i(c \circ \nabla^S) = -i\gamma^\mu \nabla^S_{\mu}$ – the Dirac operator.
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Outline

1. Introduction & motivation
2. Noncommutative geometry
3. Causal structures
4. Testing the concepts – almost commutative causality
5. Summary
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- Two points $p, q$ are **causally related** $p \preceq q$ iff $p = q$ or $\exists$ a future directed causal curve linking $p$ and $q$.

- $\preceq$ induces a partial order relation on the set of points of $M$.

- **Global hyperbolicity** $\implies$ no closed causal curves

**Theorem [Geroch (1967)]**
Compact Lorentzian manifold always contain closed causal curves.

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Gelfand - Naimark theorem [1943]

commutative $C^*$-algebras $\overset{1:1}{\leftrightarrow}$ (locally) compact Hausdorff topological spaces

- States $S(A) = \{\varphi\}$ on $A$:
  - positive linear functionals with $\|\varphi\| = 1$
  - $S(A)$ is a closed convex set
  - $P(A)$ - extremal points - pure states

- Connes (pseudo-)distance formula: (may be infinite)
  \[ d(\varphi, \chi) = \sup \{ |\varphi(a) - \chi(a)| : a \in A, \|[D, a]\| \leq 1 \}. \]

- Points of $X \overset{1:1}{\leftrightarrow} P(C(X))$ \quad $\forall x \in X$ $\varphi_x : A \to \mathbb{C}$, $\varphi_x(f) := f(x)$
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Causal functions

\[ C(M) = \{ f \in C^\infty(M, \mathbb{R}) : f \text{ is non-decreasing along future dir. causal curves} \} \]

Proposition [F. Besnard (2009)]

Let \( M \) be a globally hyperbolic Lorentzian manifold, then the set of smooth bounded causal functions \( C(M) \subset \widetilde{A} = C_b^\infty(M) \) completely determines the causal structure on \( M \) by

\[ \forall p, q \in M, \quad p \preceq q \iff \forall f \in C(M), \quad f(p) \leq f(q). \]

A causal cone \( \mathcal{C} \) is a subset of elements in \( \widetilde{A} \) such that:

(a) \( \forall a, b \in C \quad a^* = a \), \( \forall a, b \in C \quad a + b \in C \);

(c) \( \forall a \in C \quad \forall \lambda \geq 0 \quad \lambda a \in C \), \( \forall x \in \mathbb{R} \quad x1 \in C \);

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(e) \( \forall a \in \mathcal{C}, \forall \phi \in \mathcal{H}, \quad \langle \phi, \mathcal{J}[D, a]\phi \rangle \leq 0 \)
Proposition [N. Franco, M.E. (2013)]

Let $\mathcal{C}$ be a causal cone, then for every two states $\chi, \xi \in S(\tilde{\mathcal{A}})$ define

$$\chi \preceq \xi \iff \forall a \in \mathcal{C} \quad \chi(a) \leq \xi(a).$$

The relation $\preceq$ defines a partial order relation on $S(\tilde{\mathcal{A}})$.

Theorem [N. Franco, M.E. (2013)]

Let $(\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, \mathcal{J})$ be a commutative Lorentzian spectral triple constructed from a globally hyperbolic Lorentzian manifold $M$. Then,

$$P(\mathcal{A}) \simeq \text{Spec}(\mathcal{A}) \cong M,$$

and the partial order relation $\preceq$ on $S(\tilde{\mathcal{A}})$ restricted to $P(\mathcal{A})$ corresponds to the usual causal relation on $M$. 
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1. Introduction & motivation

2. Noncommutative geometry
   - Spectral triples - a reminder
   - Lorentzian spectral triples
   - Commutative examples

3. Causal structures
   - Causality - rudiments
   - Algebraisation

4. Testing the concepts – almost commutative causality
   - The “two-sheeted” space-time
   - The $M_2(\mathbb{C})$ model

5. Summary
Almost commutative flat space-time

Theorem [N. Franco, M.E. (2014a)]

Let $(\mathcal{A}_M, \tilde{\mathcal{A}}_M, \mathcal{H}_M, \mathcal{D}_M, \mathcal{J}_M)$ be an even Lorentzian spectral triple with $\mathbb{Z}_2$-grading $\gamma_M$ and a finite Riemannian spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$. Then

$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_F, \quad \tilde{\mathcal{A}} = \tilde{\mathcal{A}}_M \otimes \mathcal{A}_F, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F, \quad \mathcal{D} = \mathcal{D}_M \otimes 1 + \gamma_M \otimes \mathcal{D}_F,$$

is a Lorentzian spectral triple.

- A commutative spectral triple for Minkowski space-time
  - $\mathcal{A}_M = \mathcal{S}(\mathbb{R}^{1,n})$ - rapidly decreasing functions,
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If at least one of the $C^*$-algebras $A_1, A_2$ is commutative, then $P(A_1 \otimes A_2) \cong P(A_1) \times P(A_2)$, i.e. pure states on $A_1 \otimes A_2$ are separable.
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- We work on 2-dim Minkowski spacetime $M$.
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- $P(A_F) \cong \mathbb{Z}^2$, hence $\mathcal{M}(A_M \otimes A_F) \cong \mathbb{R}^{1,1} \cup \mathbb{R}^{1,1}$.

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Let $p \in \mathbb{R}^{1,1}_{(1)}$ and $q' \in \mathbb{R}^{1,1}_{(2)}$ then $p \preceq q'$ if and only if

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Michał Eckstein (Kraków)  
Causal Structure for NCG  
Marseille, 16th July 2014  
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  \[ \mathcal{A}_F = M_2(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^2, \quad \mathcal{D}_F = \text{diag}\{d_1, d_2\}, \text{ with } d_1 \neq d_2 \in \mathbb{R}^*. \]

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Two pure states $\omega_{p,\xi}, \omega_{q,\varphi}$ are causally related with $\omega_{p,\xi} \preceq \omega_{q,\varphi}$ if and only if:

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Outline

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2. Noncommutative geometry
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- Generalisation to higher dim, curved, more noncommutative, ... – Volunteers welcome!

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