# Geometric Unification

Ali H. Chamseddine
American University of Beirut, Lebanon
&
Institut des Hautes Etudes Scientifique (IHES)
(France)

# References

- Work in Collaboration with Alain Connes
- "Spectral Action Principle" Comm. Math. Phys. 1997
- "Gravity and the SM with neutrino mixing" Adv. Theo. Math. Phys. 2007
- "NCG as framework to unify all fundamental interactions including gravity" Fort. Phys. 2010
- "Resilience of the spectral SM" JHEP 2012

# Need for new geometry

- The large scale global picture of space-time is well described in terms of Riemannian geometry, but this picture breaks down in the high energy scales where the quantum picture takes over.

# Need for new geometry

- It is thus natural to look for a paradigm of geometry which starts from the quantum framework, where the role of real variables is played by self- adjoint operators in Hilbert space.

# Need for new geometry

- Such a framework for geometry has been slowly emerging under the name of noncommutative geometry. One of its key features, besides the ability to handle spaces for which coordinates no longer commute with each other, is that this new geometry is spectral.

# Need for new geometry

- This is in agreement with physics in which most of the data we have, either about the far distant parts of the universe or about high energy physics, are also of spectral nature.

# Need for new geometry

- The red shifted spectra of distant galaxies or the momentum eigenstates of outgoing particles in high energy experiments both point towards a prevalence of spectral nature.

# Need for new geometry

- From the mathematical standpoint it takes some doing to obtain a purely spectral

(Hilbert space theoretical) counterpart of Riemannian geometry.

One reason for the difficulty of this task is that, as is well known since the examples of

J. Milnor, non-isometric Riemannian spaces exist which have the same spectra (for the Dirac or Laplacian operators).

# Need for new geometry

- Another reason is that the conditions for a (compact) space to be a smooth manifold are given in terms of the local charts, whose existence and compatibility is assumed, but whose intrinsic meaning is more elusive.

# Need for new geometry

- The laws of physics at low energies of order of 100 Gev are well encoded by the Standard Model action (with massive neutrinos) and the Einstein-Hilbert action. The fields in the standard model are the quarks, leptons, gauge fields and a Higgs field.

# Need for new geometry

- These fields have different status than the gravitational field which depends only on the geometry of a Riemannian manifold M. The natural group of invariance of this theory is the semidirect product of the gauge transformations of U (l) x SU(2) x SU (3) and Diff (M).

# Why the Standard Model

- Many questions in the Standard Model are begging for answers, such as:
- Why the above gauge group?
- Why 16 fermions per generation?

# Why the Standard Model

- Why the particular representations of fermions?
- Why three generations?
- Why the particular Yukawa couplings and the huge hierarchy in the masses ranging from the neutrino masses to the top quark mass?

# Why the Standard Model

- Why the Higgs mechanism and spontaneous symmetry breaking?
- Is there gauge couplings unification? What determines the Higgs couplings?
- What is protecting the Higgs mass from the quadratic divergencies in what is known as the hierarchy problem?

# Why the Standard Model

- Why small left-handed neutrino masses?
- Why θ of QCD is smaller than $10^{-9}$?
- Is there new physics beyond the SM?

At present there are no compelling answers to most of these questions.

# Which geometry?

- The group *G* of symmetries of the Lagrangian of gravity coupled with matter is handed to us by physics. It is the semi-direct product of the group Map( *M*, *G*) of gauge transformations of second kind by the symmetry group of gravity, namely the group Diff( *M*) of diffeomorphisms of ordinary space-time M:

  *G* = Map( *M*, *G*) t> Diff( *M*)

# Which geometry?

- Now for gravity coupled with matter to be pure gravity on a new space *N* the most obvious requirement is to find the manifold *N* in such a way that

$$\mathrm{Diff}(N) = G$$

# Which geometry

- There is a general mathematical result which asserts that the connected component of identity in Diff ($N$) is a *simple* group for any manifold $N$. Thus, since $G$ has the non-trivial normal subgroup Map( $M$, $G$) there is no way one can solve the above equation using ordinary manifolds $N$.

# Which geometry

- One can show that there is a solution, provided one searches for noncommutative solutions. i.e. that the group *G* is indeed the group of diffeomorphisms of a new space *N*.

# Noncommutative Geometry

- The basic data is that of a spectral triple

$$(A, H, D)$$

which gives a representation in Hilbert space *H* of both the algebra *A* of coordinates and of the inverse line element *D*.

- Given a von Neumann algebra *A* of operators in Hilbert space *H* one can always find an anti-unitary isometry *J* such that the following commutators vanish :

# Noncommutative Geometry

$[x, Jy^* J^{-1}] = 0 \qquad \forall x, y \in A$

The basic rules are

- $J^2 = \varepsilon\,, \quad DJ = \varepsilon\, JD, \quad J\gamma = \varepsilon'\ \gamma J, \quad D\gamma = -\gamma D$
- $J$: charge conjugation.
- $\gamma$: chirality.

where γ is the Z/2 grading operator.

# Noncommutative Geometry

The *KO*-theory comes in 8 different versions which just depend upon the dimension of the geometry modulo 8. They are distinguished by the three possible signs $\varepsilon \in \pm 1$ which govern the algebraic rules and whose values are according to the dimension modulo 8.

# Noncommutative Geometry

In physics terms these data have the following names and meaning:

- *H*: one particle Euclidean Fermions.
- *D*: inverse propagator.

Thus the new formalism for geometry keeps a very close contact with physics. Exactly as the inner automorphisms form an "internal" part of the group of geometric symmetries, the metric admits "inner fluctuations".

# Geometry of Space-Time

- Space-time could be approximated by a noncommutative space which is a product of a continuous four-dimensional Riemannian manifold times a finite dimensional space . This space is almost commutative with the noncommutativity arising from the matrix structure of the discrete space F.

# Geometry of Space-Time

- The main intrinsic reason for crossing by a finite geometry *F* has to do with the value of the dimension of space-time modulo 8. We needed this *KO*-dimension to be 2 modulo 8 (or equivalently 10) to define the Fermionic action, since this eliminates the doubling of fermions in the Euclidean framework.

# Geometry of Space-Time

In other words the need for crossing by *F* is to shift the *KO*-dimension from 4 to 2 (modulo 8) .

This finite geometry were derived from first principles through the following steps :

# Geometry of Space-Time

- Classified the irreducible triplets

$$(A, H, J)$$

- Studied the Z/2-gradings $\gamma$ on $H$.
- Classified the subalgebras $A_F \subset A$ which allow for an operator $D$ that does not commute with the center of $A$ but fulfills the

# Classification of finite spaces

"order one" condition

$$[[D, a], b^0] = 0$$

which guarantees the linearity of the connection.

- The classification in the first step shows that the solutions fall in two classes the first of which is inconsistent with *KO*-dimension 6.

# Classification of finite spaces

For the second class, we have shown that among the very few choices of lowest dimension we obtain the case

$$A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

where $\mathbb{H}$ is the skew field of quaternions.

Note that this determines the number of fermions to be

$$4^2 = 16$$

# Classification of finite spaces

Our main result then is that there exists up to isomorphism a unique involutive subalgebra $A_F$ of maximal dimension isomorphic to

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

and together with its representation in

$$(H, J, \gamma)$$

gives the noncommutative geometry $F$ that recovers the Standard Model coupled to gravity using the spectral action

# Classification of finite spaces

- This result is remarkable because the input that was used is minimal and the first possibility obtained consistent with the axioms of noncommutative geometry, after imposing the symplectic-unitary symmetry condition on the algebra, is the algebra of the standard model with fermions in the correct representation.

# Classification of finite spaces

- All the arbitrariness that is usually encountered in the construction of the standard model whether in the choice of the $SU(3) \times SU(2) \times U(1)$ gauge group, the fermionic representations, or the Higgs structure and the electroweak spontaneous breaking mechanism, disappear .

# Classification of finite spaces

- The standard model becomes completely determined. In this respect we see that there is a geometrical structure responsible for all the details of the standard model. Geometrically we see that the underlying algebra is a direct sum of two algebras.

# Classification of finite spaces

- The first algebra is quaternionic

$$M_2(\mathbb{H})$$

broken to

$$(\mathbb{C} \oplus \mathbb{C})_R \oplus \mathbb{H}_L$$

decomposes by the chirality operator into a left-handed and right-handed sectors. The second algebra $M_4(\mathbb{C})$ is broken by the

# Classification of finite spaces

Majorana mass for right-handed neutrino into

$$\mathbb{C} \oplus M_3(\mathbb{C})$$

and corresponds to the splitting of the leptons and quarks. The fermions follow the product representation of the two algebras.

# The Spectrum

- The spectrum of the fermionic particles, which is the number of states in the Hilbert space per family is predicted to be $4^2 = 16$.
- The 16 spinors get the correct quantum number with respect to the standard model gauge group which follow the decomposition:

# The Spectrum

$(4, 4) \rightarrow$

$(1_R + 1_R + 2_L, 1 + 3) =$

$(1_R, 1) + (1_R, 1) + (2_L, 1) + (1_R, 3)$

$+ (1_R, 3) + (2_L, 3)$

These spinors correspond to $\nu_R$, $e_R$, $l_L$, $u_R$, $d_R$, $q_L$ respectively, where $l_L$ is the left-handed neutrino-electron doublet and $q_L$ is the left-handed up-down quark doublet.

# The Spectrum

- In addition to the gauge bosons of $SU(3) \times SU(2) \times U(1)$ which are the inner fluctuations of the metric along continuous directions, we also have a Higgs doublet which corresponds to the inner fluctuations of the metric along the discrete directions.
- A singlet Scalar field whose vev gives Majorana mass to the right-handed neutrinos.

# The Spectrum

- What is peculiar about the Higgs doublet, is that its mass term as determined from the spectral action comes with a negative sign and a quartic term with a plus sign, thus predicting the phenomena of spontaneous breakdown of the electroweak symmetry.

# Spectral Action

- The fermionic action takes the simple form

$$(J\Psi, D_A\Psi)$$

where $\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}$

with $\psi$ being the 16 dim spinor and $\psi^c$ the conjugate

# Spectral Action

spinor so that Ψ satisfies both Majorana and Weyl condition

$J\Psi=\Psi$ and

$\gamma\Psi=\Psi$.

In addition

$D_A=D+A+JA^*J^{-1}$

Where A contains all the gauge fields, Higgs doublet and the neutrino singlet.

# Spectral Action

The missing ingredient, in the above description of the Standard Model coupled to gravity, is provided by a simple action principle:

the spectral action principle

that has the geometric meaning of

"pure gravity"

and delivers the action functional of the Standard Model coupled to gravity when evaluated on $M \times F$.

# Spectral Action

The spectral action principle is the simple statement that

"the physical action is determined by the spectrum of the Dirac operator *D*."

The spectral data are available in localized form anywhere, and are (asymptotically) of the additive form

# Spectral Action

$S_b = \text{Trace}\,(f\,(D/\Lambda)).$

The detailed form of the function *f* is largely irrelevant since the spectral action can be expanded in decreasing powers of the scale $\Lambda$ and the function *f* only appears through the scalars

$$f_k = \int_0^\infty f(y)\,y^{k-1}\,dy$$

# Spectral Action (cont.)

$$
\begin{aligned}
S_{\mathrm{b}} = \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x \sqrt{g} \qquad\qquad (5.49) \\
- \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x \sqrt{g} \left( R + \frac{1}{2} a \overline{H} H + \frac{1}{4} c \sigma^2 \right) \\
+ \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g} \left[ \frac{1}{30} \left( -18 C^2_{\mu\nu\rho\sigma} + 11 R^* R^* \right) + \frac{5}{3} g_1^2 B^2_{\mu\nu} + g_2^2 \left( W^{\alpha}_{\mu\nu} \right)^2 + g_3^2 \left( V^m_{\mu\nu} \right)^2 \right. \\
\left. + \frac{1}{6} a R \overline{H} H + b \left( \overline{H} H \right)^2 + a \left| \nabla_\mu H_a \right|^2 + 2e \overline{H} H \, \sigma^2 + \frac{1}{2} d \, \sigma^4 + \frac{1}{12} c R \sigma^2 + \frac{1}{2} c \left( \partial_\mu \sigma \right)^2 \right] \\
+ F_{-2} \Lambda^{-2} a_6 + \cdots
\end{aligned}
$$

# Spectral Action

$$
\begin{aligned}
S_{\mathrm{f}} = {} & \nu_R^* \gamma^\mu D_\mu \nu_R \\
& + e_R^* \gamma^\mu \left( D_\mu + i g_1 B_\mu \right) e_R \\
& + l_L^{a*} \gamma^\mu \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta_a^b - \frac{i}{2} g_2 W_\mu^\alpha \left( \sigma^\alpha \right)_a^b \right) l_{bL} \\
& + u_R^{i*} \gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left( \lambda^m \right)_i^j \right) u_{jR} \\
& + d_R^{i*} \gamma^\mu \left( \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left( \lambda^m \right)_i^j \right) d_{jR} \\
& + q_L^{ia*} \gamma^\mu \left( \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta_a^b \delta_i^j - \frac{i}{2} g_2 W_\mu^\alpha \left( \sigma^\alpha \right)_a^b \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left( \lambda^m \right)_i^j \delta_a^b \right) q_{jbL} \\
& + \nu_R^* \gamma_5 k^{*\nu} \epsilon^{ab} H_b l_{aL} + e_R^* \gamma_5 k^{*e} \overline{H}^a l_{aL} \\
& + u_R^{i*} \gamma_5 k^{*u} \epsilon^{ab} H_b \delta_i^j q_{jaL} + d_R^{i*} \gamma_5 k^{*d} \overline{H}^a \delta_i^j q_{jaL} + \nu_R^* \gamma_5 k^{*\nu_R} \sigma \left( \nu_R^* \right)^c + \mathrm{h.c}
\end{aligned}
$$

# Singlet Role

- The presence of the singlet insures that the Higgs coupling does not turn negative at high energies. The potential takes the form, after rescaling

$$V = \frac{1}{4}\left(\lambda_h \overline{h}^4 + 2\lambda_{h\sigma}\overline{h}^2\overline{\sigma}^2 + \lambda_\sigma \overline{\sigma}^4\right) - \frac{2g^2}{\pi^2} f_2 \Lambda^2 \left(\overline{h}^2 + \overline{\sigma}^2\right)$$

where

$$\lambda_h = \frac{n^2+3}{(n+3)^2}\left(4g^2\right)$$
$$\lambda_{h\sigma} = \frac{2n}{n+3}\left(4g^2\right)$$
$$\lambda_\sigma = 2\left(4g^2\right)$$

# Prediction of the Spectral SM

- We have classified all noncommutative spaces formed as a product of a continuous four dimensional space times a discrete space consistent with the axioms of noncommutative geometry.

# Prediction of the Spectral SM

- Under the very weak physical assumptions that there are no mirror fermions in nature and the existence of a fermionic Majorana mass, we have determined uniquely that the resulting noncommutative space corresponds to the spectrum of the Standard Model of particle interactions in addition to right-handed neutrinos and an extra singlet scalar field.

# Prediction of the Spectral SM

- We have predicted that the number of fermions per family is 16 with the correct representations under the symmetry U(1)xSU(2)xSU(3).
- Explained the extremely small mass of the neutrino through the see-saw mechanism.

# Prediction of the Spectral SM (cont.)

- Determined the existence of a Higgs doublet as the fluctuation of the metric (Dirac operator) along discrete directions, and the spontaneous symmetry breaking mechanism and the origin of the negative mass term in potential.

# Prediction of the Spectral SM

- The existence of a new scalar field, whose vev gives a Majorana mass to the right-handed neutrinos, and which is essential in protecting the Higgs coupling from turning to negative at energies of order $10^{11}$ Gev. This paves the way for the Standard Model to hold all the way up to very high energies.

# Prediction of the Spectral SM

- Predicted the top quark mass to be of the order of 170 Gev. (and thus no fourth generation)
- Provided a geometrical framework for the unification of all particle interactions including gravity valid up to very high energies.

# Prediction of the Spectral SM

- Replaced diffeomorphism invariance in general relativity and gauge symmetry of vector bosons with outer and inner automorphisms of the algebra defining the noncommutative space.

# Prediction of the Spectral SM

- Euclidean formulation of quantum gravity requires the addition of the Hawking-Gibbons boundary term to the Einstein action. The spectral action contains the Hawking-Gibbons term automatically, reflecting the fact that

# Prediction of the Spectral SM

- noncommutative geometry being dependent on the Dirac operator, which is the inverse of the fermion propagator, contains information about quantum gravity.

$$-\frac{1}{16\pi}\int_M d^4x\sqrt{g}R-\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}K,$$

# Predictions of the SM

- In the spectral action, because of the left-right symmetries, all parity violating terms, except for those coming from the weak sector

$$G(W^a_{\mu\nu})^2$$

are absent, in particular the theta term of QCD

$$\theta(V^i_{\mu\nu})^2$$

is absent at the tree level, and under one condition can be shown to remain so to order $10^{-9}$.

# Beyond SM

- Removing one of the conditions of noncommutative geometry which guarantees the linearity of the connection allows for inner fluctuations which contains a quadratic dependence. These fluctuations have the very interesting property of forming a semi-group (i.e. containing non-invertible elements).

$$D_A = D + A + JA^* J^{-1} + A_{(2)}$$

# Beyond SM

- The underlying symmetry in this case is then

extended to the Pati-Salam unification group

$$SU(2)_R x SU(2)_L x SU(4)$$

where the lepton number is the fourth color.

# Beyond SM

- The $A_{(2)}$ part of the connection depend on scalar fields, the Higgs fields that break the high energy sector that breaks

$$SU(2)_R \times SU(4)$$

to $$U(1) \times SU(3).$$

Depending on the initial structure of D without fluctuations the Higgs fields could contain parts depending quadratically on the fundamental Higgs fields or contain fields in the product representations.

# Beyond SM

- The simplest Higgs fields structure is
- $(2_R,2_L,1)+(2_R,1_L,4)+(1_R,1_L,1+15)$.
- The Higgs potential and interactions are fixed unambiguously.
- Symmetry breaking occurs at GUT scale and truncates to the SM at low energies.

# Beyond SM

- The Standard Model continues to hold experimentally to a very high degree of precision:

In the noncommutative setting, any deviation from the Standard Model can only be consistent with the Pati-Salam model with a connection containing quadratic dependence.

# Conclusions

- Noncommutative geometry as developed by Alain Connes provides an attractive geometric setting for the unification of all fundamental interactions including gravity.
- For finite spaces consistent with the linearity of the connection, the SM is singled out in a unique way, extended by right-handed neutrinos and a singlet.

# Conclusions

- Noncommutative geometry incorporates the language of Quantum mechanics provides a natural framework to unify all fundamental interactions and and has so far been successful in understanding many of the whys of the standard model.

# Conclusions

- NCG provides promising directions of research on problems related to quantization of field theories in general and gravity in particular.
- It provides explanations to many of the questions that have no answers in the SM.
- Any future deviations from the SM can only be due to a Pati-Salam unification with a very well defined structure.

# Conclusions

- Many interesting developments are still needed such as a quantizing scheme that takes the noncommutative geometric constraints into account.
- Many questions answered, but many remain to be answered.
- Details given on Video in IHES Course of four lectures (also on Youtube), June 2014.