Noncommutative Geometry in Physics

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- Introduction
- A Brief Summary of AC NCG
- Noncommutative Space of SM
- Spectral Action Principle
- Spectral Action for NC Space with Boundary
- Beyond the Standard Model

- Based on collaborative work with **Alain Connes** in publications: *

- *The Spectral Action Principle, Comm. Math. Phys. 186, 731-750 (1997)*

- *Scale Invariance in the Spectral Action, J. Math. Phys. 47, 063504 (2006)*

- *Noncommutative Geometry as a framework to unify all fundamental interactions, For. Phys.58 (2010) 53. .*

- *Boundary Terms in Quantum Gravity from Spectral Action of Noncommutative Space, Phys. Rev. Lett. 99 071302 (2007).*

- *Why the Standard Model Journ. Geom. Phys.* 58:38-47,2008.

- *Resilience of the Spectral Standard Model, JHEP 1209 (2012)104*

- *Beyond the Spectral Standard Model, JHEP 1311 (2013) 132 (also with W. van Suijlekom).*

- *IHES Course on Video, Four lectures, June 2014.*

1 Introduction

- Taking GR as prototype for other forces where Geometry determines the dynamics, we will set to construct geometrical spaces and associate with these dynamical actions.
- Dirac operator is a basic ingredient in defining noncommutative spaces.
- Eigenvalues of Dirac operators define geometric invariants. The Spectral action is a function of these eigenvalues.
- The only restriction on the function is that it is a positive function.
- Principle although simple works in a large number of cases.

2 *A Brief Summary of AC NCG*

The basic idea is based on physics. The modern way of measuring distances is spectral. The units of distance is taken as the wavelength of atomic spectra. To adopt this geometrically we have to replace the notion of real variable which one takes as a function f on a set X, $f : X \to R$. It is now given by a self adjoint operator in a Hilbert space as in quantum mechanics.

The space X is described by the algebra $\mathcal{A}$ of coordinates which is represented as operators in a fixed Hilbert space $\mathcal{H}$.The geometry of the noncommutative space is determined in terms of the spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J\ , \gamma)$. A real, even spectral triple is defined by

- $\mathcal{A}$ an associative algebra with unit 1 and involution $*$.
- $\mathcal{H}$ is a complex Hilbert space carrying a faithful representation π of the algebra.
- $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ with the resolvent $(D - \lambda)^{-1}, \lambda \notin \mathbf{R}$ of D compact.
- J is an anti–unitary operator on $\mathcal{H}$, a real structure (charge conjugation.)
- γ is a unitary operator on $\mathcal{H}$, the chirality.

We require the following axioms to hold:

- $J^2 = \epsilon$, ($\varepsilon = 1$ in zero dimensions and $\varepsilon = -1$ in 4 dimensions).

- $[a, b^o] = 0$ for all $a, b \in \mathcal{A}$, $b^o = Jb^*J^{-1}$. This is the zeroth order condition. This is needed to define the right action on elements of $\mathcal{H}$: $\zeta b = b^o \zeta$.

- $DJ = \varepsilon' JD, \quad J\gamma = \varepsilon'' \gamma J, \quad D\gamma = -\gamma D$ where $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$. The reality conditions resemble the conditions of existence of Majorana (real) fermions.

- $[[D, a], b^o] = 0$ for all $a, b \in \mathcal{A}$. This is the first order condition.

- $\gamma^2 = 1$ and $[\gamma, a] = 0$ for all $a \in \mathcal{A}$. These properties allow the decomposition

  $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$.

- $\mathcal{H}$ is endowed with $\mathcal{A}$ bimodule structure $a\zeta b = ab^o\zeta$.

- The notion of dimension is governed by growth of eigenvalues, and may be fractals or complex.

- $\mathcal{A}$ has a well defined unitary group

$$\mathcal{U} = \{u \in \mathcal{A}; \quad u\,u^* = u^*u = 1\}$$

  The natural adjoint action of of $\mathcal{U}$ on $\mathcal{H}$ is given by $\zeta \to u\zeta u^* =$

$u\,J\,u\,J^*\zeta \quad \forall \zeta \in \mathcal{H}$. Then

$$\langle \zeta, D\zeta \rangle$$

is not invariant under the above transformation:

$$(u\,J\,u\,J^*)\,D\,(u\,J\,u\,J^*)^* = D + u\,[D, u^*] + J\,(u\,[D, u^*])\,J^*$$

- Then the action $\langle \zeta, D_A \zeta \rangle$ is invariant where

$$D_A = D + A + \varepsilon' JAJ^{-1}, \quad A = \sum_i a^i\,[D, b^i]$$

and $A = A^*$ is self-adjoint. This is similar to the appearance of the interaction term for the photon with the electrons

$$i\overline{\psi}\gamma^\mu \partial_\mu \psi \to i\overline{\psi}\gamma^\mu\,(\partial_\mu + ieA_\mu)\,\psi$$

to maintain invariance under the variations

$\psi \to e^{i\alpha(x)}\psi$.

- A real structure of KO-dimension $n \in \mathbb{Z}/8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that

$$J^{\,2} = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\,\gamma = \varepsilon'' \gamma J \text{ (even case)}.$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of n mod 8 given by

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- The algebra $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space. The spectral geometry of $\mathcal{A}$ is given by the product rule $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is finite dimensional, and

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F, \quad D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$

where $L^2(M, S)$ is the Hilbert space of L^2 spinors, and D_M is the Dirac operator of the Levi-Civita spin connection on M, $D_M = \gamma^\mu (\partial_\mu + \omega_\mu)$. The Hilbert space $\mathcal{H}_F$ is taken to include the physical fermions. The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$.

In order to avoid the fermion doubling problem $\zeta, \zeta^c, \zeta^*, \zeta^{c*}$ where $\zeta \in \mathcal{H}$, are not independent) it was shown that the finite dimensional space must be taken to be of K-theoretic dimension 6 where in this case $(\varepsilon, \varepsilon', \varepsilon") = (1, 1, -1)$ (so as to impose the condition $J\zeta = \zeta$). This makes the total K-theoretic dimension of the noncommutative space to be 10 and would allow to impose the reality (Majorana) condition and the Weyl condition simultaneously in the Minkowskian continued form, a situation very familiar in ten-dimensional supersymmetry. In the Euclidean version, the use of the J in the fermionic action, would give for the chiral fermions in the path integral, a Pfaffian instead of determinant, and will thus cut the fermionic degrees of freedom by 2. In other words, to have the fermionic sector free of the fermionic doubling problem we must make the choice

$$J_F^2 = 1, \qquad J_F D_F = D_F J_F, \qquad J_F \gamma_F = -\gamma_F J_F$$

In what follows we will restrict our attention to determination of the finite algebra, and will omit the subscript F.

3 Noncommutative Space of Standard Model

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- There are two main constraints on the algebra from the axioms of noncommutative geometry. We first look for involutive algebras $\mathcal{A}$ of operators in $\mathcal{H}$ such that,

$$[a, b^0] = 0\,, \quad \forall\, a, b \in \mathcal{A}\,.$$

where for any operator a in $\mathcal{H}$, $a^0 = Ja^*J^{-1}$. This is called the order zero condition. We shall assume that the following two conditions to hold. We assume the representation of $\mathcal{A}$ and J in $\mathcal{H}$ is *irreducible.*

- Classify the irreducible triplets $(\mathcal{A}, \mathcal{H}, J)$.

- In this case we can state the following theorem: *The center* $Z(\mathcal{A}_{\mathbb{C}})$ *is* C *or* $C \oplus C$.

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C}$ then $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C})$ and $\mathcal{A} = M_k(\mathbb{C})\,,\ M_k(\mathbb{R})$ and $M_a(\mathbb{H})$ for even $k = 2a$, where $\mathbb{H}$ is the field of quaternions. These correspond respectively to the unitary, orthogonal and symplectic case. The dimension of $\mathcal{H}$ Hilbert spac is $n = k^2$ is a square and $J(x) = x^*, \quad \forall x \in M_k(\mathbb{C})\,.$

- If the center $Z(\mathcal{A}_{\mathbb{C}})$ is $\mathbb{C} \oplus \mathbb{C}$ then we can state the theorem: *Let* H *be a Hilbert space of dimension* n. *Then an irreducible solution with*

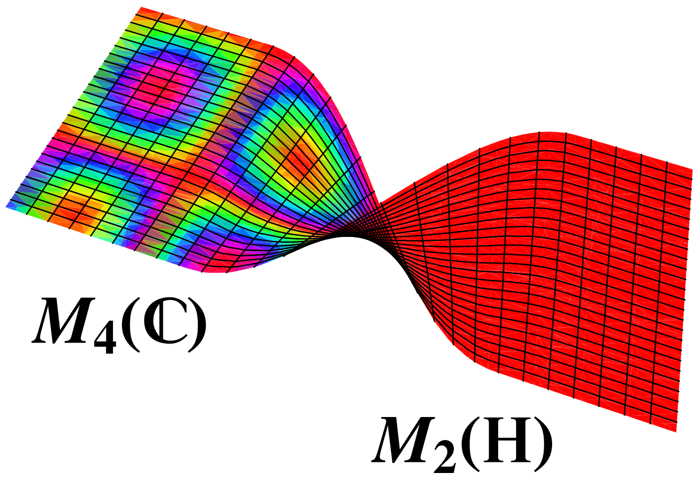
$Z(\mathcal{A}_{\mathbb{C}}) = C \oplus C$ exists iff $n = 2k^2$ is twice a square. It is given by $A_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution

$$J(x, y) = (y^*, x^*), \quad \forall x, y \in M_k(\mathbb{C}).$$

With each of the $M_k(\mathbb{C})$ in $\mathcal{A}_{\mathbb{C}}$ we can have the three possibilities $M_k(\mathbb{C})$, $M_k(\mathbb{R})$, or $M_a(\mathbb{H})$, where $k = 2a$. At this point we make the *hypothesis* that we are in the "symplectic–unitary" case, thus restricting the algebra $\mathcal{A}$ to the form $\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})$, $k = 2a$. The dimension of the Hilbert space $n = 2k^2$ then corresponds to k^2 fundamental fermions, where $k = 2a$ is an even number. The first possible value for k is 2 corresponding to a Hilbert space of four fermions and an algebra $\mathcal{A} = \mathbb{H} \oplus M_2(\mathbb{C})$. The existence of quarks rules out this possibility. The next possible value for k is 4 predicting the number of fermions to be 16.

Up to an automorphisms of A^{ev}, there exists a unique involutive subalgebra $A_F \subset A^{\mathrm{ev}}$ of maximal dimension admitting off-diagonal Dirac operators

$$\mathcal{A}_F = \{\lambda \oplus q, \lambda \oplus m \,|\, \lambda \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})\}$$
$$\subset \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$$



isomorphic to $C \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

We denote the spinors as follows

$$\psi_A = \psi_{\alpha I} = (\psi_{\alpha 1}, \psi_{\alpha i})$$
$$= \left(\psi_{\dot{1}1}, \psi_{\dot{2}1}, \psi_{a1}, \psi_{\dot{1}i}, \psi_{\dot{2}i}, \psi_{ai}\right)$$
$$\equiv (\nu_R, e_R, l_a, u_{Ri}, d_{Ri}, q_{ai})$$

where $l_a = (\nu_L, e_L)$ and $q_{ai} = (u_{Li}, d_{Li})$. The component $\psi_{\dot{1}'1'} = \psi^c_{\dot{1}1}$ so that we get

$$\psi^*_A D^B_A \psi_B + \nu^{*c}_R k^{*\nu_R} \nu_R + cc$$

Needless to say the term $\psi^*_A D^B_A \psi_B$ contains all the fermionic interaction terms in the standard model.

Write the Dirac operator in the form

$$D = \begin{pmatrix} D_A^B & D_A^{B'} \\ D_{A'}^B & D_{A'}^{B'} \end{pmatrix},$$

where

$$A = \alpha I, \quad \alpha = 1, \cdots, 4, \quad I = 1, \cdots, 4$$

$$A' = \alpha' I', \quad \alpha' = 1', \cdots, 4', \quad I = 1', \cdots, 4'$$

Thus $D_A^B = D_{\alpha I}^{\beta J}$. We start with the algebra

$$\mathcal{A} = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$$

and write

$$a = \begin{pmatrix} X_\alpha^\beta \delta_I^J & 0 \\ 0 & \delta_{\alpha'}^{\beta'} Y_{I'}^{J'} \end{pmatrix}$$

In this form

$$a^o = J a^* J^{-1} = \begin{pmatrix} \delta_\alpha^\beta Y_I^{tJ} & 0 \\ 0 & X_{\alpha'}^{*\beta'} \delta_{I',}^{J'} \end{pmatrix}$$

and clearly satisfy $[a, b^o] = 0$. The order one condition is

$$[[D, a], b^o] = 0$$

Write

$$b^o = \begin{pmatrix} \delta_\alpha^\beta W_I^J & 0 \\ 0 & Z_{\alpha'}^{\beta'} \delta_{I'}^{J'} \end{pmatrix}$$

then

$$\begin{aligned} &[[D,a],b^o] \\ &= \begin{pmatrix} \left[\left[D_A^B, X\right], W\right] & \left(D_A^{B'}Y - XD_A^{B'}\right)Z - W\left(D_A^{B'}Y - XD_A^{B'}\right) \\ \left(D_{A'}^B X - YD_{A'}^B\right)W - Z\left(D_{A'}^B X - YD_{A'}^B\right) & \left[\left[D_{A'}^{B'}, Y\right], Z\right] \end{pmatrix} \end{aligned}$$

Explicitely the first two equations:

$$\left(D_{\alpha I}^{\gamma K} X_\gamma^\beta - X_\alpha^\gamma D_{\gamma I}^{\beta K}\right) W_K^J - W_I^K \left(D_{\alpha K}^{\gamma J} X_\gamma^\beta - X_\alpha^\gamma D_{\gamma K}^{\beta J}\right) = 0$$

$$\left(D_{\alpha I}^{\gamma' K'} Y_{K'}^{J'} - X_\alpha^\gamma D_{\gamma I}^{\gamma' K}\right) Z_{\gamma'}^{\beta'} - W_I^K \left(D_{\alpha K}^{\beta' K'} Y_{K'}^{J'} - X_\alpha^\gamma D_{\gamma K}^{\beta' J'}\right) = 0$$

We have shown that the only solution of the second equation is

$$D_{\alpha I}^{\beta' K'} = \delta_\alpha^{\dot{1}} \delta_{\dot{1}'}^{\beta'} \delta_I^1 \delta_{1'}^{K'} k^{*\nu_R}$$

and this implies that

$$D_{\alpha I}^{\beta J} = D_{\alpha(l)}^\beta \delta_I^1 \delta_1^J + D_{\alpha(q)}^\beta \delta_I^i \delta_j^J \delta_i^j$$

$$Y_{I'}^{J'} = \delta_{I'}^{1'} \delta_{1'}^{J'} Y_{1'}^{1'} + \delta_{I'}^{i'} \delta_{j'}^{J'} Y_{i'}^{j'}$$

$$X_{\dot{1}}^{\dot{1}} = Y_{1'}^{1'}, \; X_{\dot{1}}^\alpha = 0, \quad \alpha \neq \dot{1}$$

We will be using the notation

$$\alpha = \dot{1}, \dot{2}, a \quad \text{where } a = 1, 2$$

From the property of commutation of the grading operator

$$g_\alpha^\beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

$$[g, a] = 0 \quad a \in M_4(\mathbb{C})$$

the algebra $M_4(\mathbb{C})$ reduces to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. We further impose the condition of symplectic isometry on $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$

$$\sigma_2 \otimes 1_2 (\overline{a}) \sigma_2 \otimes 1_2 = a$$

reduces it to $\mathbb{H} \oplus \mathbb{H}$. Together with the above condition this implies that

$$X_\alpha^\beta = \delta_\alpha^{\dot{1}} \delta_{\dot{1}}^\beta X_{\dot{1}}^{\dot{1}} + \delta_\alpha^{\dot{2}} \delta_{\dot{2}}^{\beta'} \overline{X}_{\dot{1}}^{\dot{1}} + \delta_\alpha^a \delta_b^\beta X_a^b$$

and the algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ reduces to

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

because $X_{\dot{1}}^{\dot{1}} = Y_{1'}^{1'}$.

With this we can form the Dirac operator of the product space of this discrete space times a four-dimensional Riemannian manifold

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F$$

Since D_F is a 32×32 matrix tensored with the 3×3 matrices of generation space, D is 384×384 matrix.

Next we have to evaluate the operator

$$D_A = D + A + JAJ^{-1}$$

where

$$A = \sum a \left[D, b \right]$$

or in tensor notation

$$A_A^B = \sum a_A^C \left(D_C^D b_D^B - b_C^D D_D^B \right)$$

(there are no mixing terms like $D_C^{D'} b_{D'}^B$ because b is block diagonal).

Writing all components of the the full Dirac operator $D_{\alpha I}^{\beta J}$

$$(D)_{\dot{1}1}^{\dot{1}1} = \gamma^\mu \otimes D_\mu \otimes 1_3, \quad D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{cd}(e)\,\gamma_{cd}, \quad 1_3 = \text{generations}$$

$$(D)_{\dot{1}1}^{a1} = \gamma_5 \otimes k^{*\nu} \otimes \epsilon^{ab} H_b \qquad k^\nu = 3\times 3 \text{ neutrino mixing matrix}$$

$$(D)_{\dot{2}1}^{\dot{2}1} = \gamma^\mu \otimes (D_\mu + i g_1 B_\mu) \otimes 1_3$$

$$(D)_{\dot{2}1}^{a1} = \gamma_5 \otimes k^{*e} \otimes \overline{H}^a$$

$$(D)_{a1}^{\dot{1}1} = \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \overline{H}^b$$

$$(D)_{a1}^{\dot{2}1} = \gamma_5 \otimes k^e \otimes H_a$$

$$(D)_{a1}^{b1} = \gamma^\mu \otimes \left(\left(D_\mu + \frac{i}{2} g_1 B_\mu\right)\delta_a^b - \frac{i}{2} g_2 W_\mu^\alpha \left(\sigma^\alpha\right)_a^b\right) \otimes 1_3, \qquad \sigma^\alpha = \text{Pauli}$$

$$(D)_{\dot{1}i}^{\dot{1}j} = \gamma^\mu \otimes \left(\left(D_\mu - \frac{2i}{3} g_1 B_\mu\right)\delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j\right) \otimes 1_3, \qquad \lambda^i = \text{Gell-Mann}$$

$$(D)_{\dot{1}i}^{aj} = \gamma_5 \otimes k^{*u} \otimes \epsilon^{ab} H_b \delta_i^j$$

$$(D)_{\dot{2}i}^{\dot{2}j} = \gamma^\mu \otimes \left(\left(D_\mu + \frac{i}{3} g_1 B_\mu\right)\delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j\right) \otimes 1_3$$

$$(D)_{\dot{2}i}^{aj} = \gamma_5 \otimes k^{*d} \otimes \overline{H}^a \delta_i^j$$

$$(D)_{ai}^{bj} = \gamma^\mu \otimes \left(\left(D_\mu - \frac{i}{6} g_1 B_\mu\right)\delta_a^b \delta_i^j - \frac{i}{2} g_2 W_\mu^\alpha \left(\sigma^\alpha\right)_a^b \delta_i^j - \frac{i}{2} g_3 V_\mu^m \left(\lambda^m\right)_i^j \delta_a^b\right) \otimes 1_3$$

$$(D)_{ai}^{\dot{1}j} = \gamma_5 \otimes k^u \otimes \epsilon_{ab} \overline{H}^b \delta_i^j$$

$$(D)_{ai}^{\dot{2}j} = \gamma_5 \otimes k^d \otimes H_a \delta_i^j$$

$$(D)_{\dot{1}1}^{\dot{1}'1'} = \gamma_5 \otimes k^{*\nu_R} \sigma \qquad \text{generate scale } M_R \text{ by } \sigma \to M_R$$

$$(D)_{\dot{1}'1'}^{\dot{1}1} = \gamma_5 \otimes k^{\nu_R} \sigma$$

$$D_{A'}^{B'} = \overline{D}_A^B$$

where the matrix form would look like

$$
\begin{pmatrix}
\dot{1}1 & \dot{2}1 & a1 & \dot{1}i & \dot{2}i & ai \\
v_R & e_R & l_a & u_{iR} & d_{iR} & q_{iL}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\dot{1}1 \\ \dot{2}1 \\ b1 \\ \dot{1}j \\ \dot{2}j \\ bj
\end{pmatrix}
\begin{pmatrix}
(D)^{\dot{1}1}_{\dot{1}1} & 0 & (D)^{a1}_{\dot{1}1} & 0 & 0 & 0 \\
0 & (D)^{\dot{2}1}_{\dot{2}1} & (D)^{a1}_{\dot{2}1} & 0 & 0 & 0 \\
(D)^{\dot{1}1}_{b1} & (D)^{\dot{2}1}_{b1} & (D)^{b1}_{a1} & 0 & 0 & 0 \\
0 & 0 & 0 & (D)^{\dot{1}i}_{\dot{1}j} & 0 & (D)^{ai}_{\dot{1}j} \\
0 & 0 & 0 & 0 & (D)^{\dot{2}i}_{\dot{2}j} & (D)^{ai}_{\dot{2}j} \\
0 & 0 & 0 & (D)^{\dot{1}i}_{bj} & (D)^{\dot{2}i}_{bj} & (D)^{ai}_{bj}
\end{pmatrix}
$$

4 *The Spectral Action Principle (SAP)*

- There is a shift of point of view in NCG similar to Fourier transform, where the usual emphasis on the points on the points $x \in M$ of a geometric space is now replaced by the spectrum Σ of the operator D. The existence of Riemannian manifolds which are isospectral but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity

The physical action depends only on the Σ

This is the spectral action principle (SAP) . The spectrum is a geometric invariant and replaces diffeomorphism invariance.

- Apply this basic principle to the noncommutative geometry defined by the spectrum of the standard model to show that the dynamics of all the interactions, including gravity is given by the spectral action

$$\text{Trace } f\left(\frac{D_A}{\Lambda}\right) + \frac{1}{2}\langle J\psi, D_A\psi\rangle$$

where f is a test function, Λ a cutoff scale and ψ represents the fermions.

- The function f only plays a role through its momenta f_0, f_2, f_4 where

$$f_k = \int_0^\infty f(v) v^{k-1} dv, \quad \text{for } k > 0, \;\; f_0 = f(0).$$

These will serve as three free parameters in the model. $S_\Lambda[D_A]$ is the number of eigenvalues λ of D_A counted with their multiplicities such that $|\lambda| \leq \Lambda$.

To illustrate how this comes, expand the function f in terms of its Laplace transform

$$\text{Trace} f\left(P\right) = \sum_s f_{s'} \text{Trace}\left(P^{-s}\right)$$

$$\text{Trace}\left(P^{-s}\right) = \frac{1}{\Gamma\left(s\right)} \int_0^\infty t^{s-1} \text{Trace}\left(e^{-tP}\right) dt \qquad \text{Re}\left(s\right) \geq 0$$

$$\text{Trace}\left(e^{-tP}\right) \simeq \sum_{n \geq 0} t^{\frac{n-m}{d}} \int_M a_n\left(x, P\right) dv\left(x\right)$$

Gilkey gives generic formulas for the Seeley-deWitt coefficients $a_n\left(x, P\right)$ for a large class of differential operators P.

- The bosonic part gives an action that unifies gravity with $SU(2) \times U(1) \times SU(3)$ Yang-Mills gauge theory, with a Higgs doublet ϕ and spontaneous symmetry breaking, in addition to a singlet that gives mass to the right-handed neutrino. It is given by

$$
\begin{aligned}
S_{\mathrm{b}} = & \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x \sqrt{g} \\
& - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x \sqrt{g} \left(R + \frac{1}{2} a \overline{H} H + \frac{1}{4} c \sigma^2 \right) \\
& + \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g} \left[\frac{1}{30} \left(-18 C^2_{\mu\nu\rho\sigma} + 11 R^* R^* \right) + \frac{5}{3} g_1^2 B^2_{\mu\nu} + g_2^2 \left(W^\alpha_{\mu\nu} \right)^2 + g_3^2 \left(V^m_{\mu\nu} \right)^2 \right. \\
& \qquad \left. + \frac{1}{6} a R \overline{H} H + b \left(\overline{H} H \right)^2 + a \left| \nabla_\mu H_a \right|^2 + 2 e \overline{H} H \, \sigma^2 + \frac{1}{2} d \, \sigma^4 + \frac{1}{12} c R \sigma^2 + \frac{1}{2} c \left(\partial_\mu \sigma \right)^2 \right] \\
& + F_{-2} \Lambda^{-2} a_6 + \cdots
\end{aligned}
$$

The Higgs-singlet potential reduces to (after some scalings)

$$V=\frac{1}{4}\left(\lambda_h\overline{h}^4+2\lambda_{h\sigma}\overline{h}^2\overline{\sigma}^2+\lambda_\sigma\overline{\sigma}^4\right)-\frac{2g^2}{\pi^2}f_2\Lambda^2\left(\overline{h}^2+\ \overline{\sigma}^2\right)$$

where

$$\begin{aligned}\lambda_h &= \frac{n^2+3}{(n+3)^2}\left(4g^2\right)\\ \lambda_{h\sigma} &= \frac{2n}{n+3}\left(4g^2\right)\\ \lambda_\sigma &= 2\left(4g^2\right)\\ n &= \left|\frac{k^\nu}{k^t}\right|^2\end{aligned}$$

The singlet has a strong coupling $\lambda_\sigma = 8g^2$. The coupling $\lambda_{h\sigma}$ vanishes for $n = 0$ and increases to $8g^2$ as $n \to \infty$. The coupling λ_h decreases from $\frac{4}{3}g^2$ to g^2 for n varying from 0 to 1 and increases again to $4g^2$ for $n \to \infty$. The condition to have a stable Higgs mass at 125 Gev is that the determinant of the mass matrix is positive implies

$$\lambda_{h\sigma}^2 < \lambda_h\lambda_\sigma \tag{1}$$

which is satisfied provided $n^2 < 3$. The physical states are mixtures of the fields h and σ but with very small mixing of order of $\frac{\overline{v}}{\overline{w}} = \mathrm{O}\left(10^{-9}\right)$.

Thus, we can change possible starting points for both n and g to hold at some unification scale. The physical masses of the top and Higgs fields are then determined from the values of the couplings at low energies:

$$m_t\left(0\right) = k^t\left(0\right)\frac{246}{\sqrt{2}} \tag{2}$$

$$m_h\left(0\right) = 246\sqrt{2\lambda_h\left(0\right)\left(1-\frac{\lambda_{h\sigma}^2(0)}{\lambda_h(0)\lambda_\sigma(0)}\right)} \tag{3}$$

Numerical studies of this system of one loop RG equations for various starting points of the parameters n, g, and unification scale reveal that a Higgs mass of around 125.5 Gev and a top quark mass of around 173 Gev.

We have now answered the following:

- Why the specific $U(1) \times SU(2) \times SU(3)$ gauge group.
- Why the particular representations.
- Why 16 fermions in one generation.
- Why one Higgs field doublet and the spontaneous symmetry breaking.
- Why the Higgs mass, the fermion masses.
- The see-saw mechanism and the smallness of the neutrino mass.

- Stability of the Standard Model up to very high energies and the existence of the singlet.

- A top quark mass of around 173 Gev and consistency of a Higgs mass of 125 Gev.

5 Spectral Action for NC Spaces with Boundary

In the Hamiltonian quantization of gravity it is essential to include boundary terms in the action as this allows to define consistently the momentum conjugate to the metric. This makes it necessary to modify the Einstein-Hilbert action by adding to it a surface integral term so that the variation of the action is well defined. The reason for this is that the curvature scalar R contains second derivatives of the metric, which are removed after integrating by parts to obtain an action which is quadratic in first derivatives of the metric. To see this note that the curvature $R \sim \partial\Gamma + \Gamma\Gamma$ where $\Gamma \sim g^{-1}\partial g$ has two parts, one part is of second order in derivatives of the form $g^{-1}\partial^2 g$ and the second part is the square of derivative terms of the form $\partial g \partial g.$ To define the conjugate momenta in the Hamiltonian formalism, it is necessary to integrate by parts the term $g^{-1}\partial^2 g$ and change it to the form $\partial g \partial g$. These surface terms, which turned out to be very important, are canceled by modifying the Euclidean action to

$$I = -\frac{1}{16\pi}\int\limits_{M} d^4x\sqrt{g}R - \frac{1}{8\pi}\int\limits_{\partial M} d^3x\sqrt{h}K,$$

where ∂M is the boundary of M, h_{ab} is the induced metric on ∂M and K is the trace of the second fundamental form on ∂M. Notice that there is a relative factor of 2 between the two terms, and that the boundary term has to be completely fixed. This is a delicate fine tuning and is not determined by any symmetry, but only by the consistency requirement. There is no known symmetry that predicts this combination and it is always added by hand. In contrast we can compute the spectral action for manifolds with boundary. The hermiticity of the Dirac operator

$$(\psi|\, D\psi) = (D\psi|\, \psi)$$

is satisfied provided that $\pi_{-}\psi|_{\partial M} = 0$ where $\pi_{-} = \frac{1}{2}(1-\chi)$ is a projection operator on ∂M with $\chi^2 = 1$. To compute the spectral action for manifolds with boundary we have to specify the condition $\pi_{-} D\psi|_{\partial M} = 0.$ The result of the computation gives the remarkable result that the Gibbons-Hawking boundary term is generated without any fine tuning. Adding matter interactions, does not alter the relative sign and coefficients of these two terms, even when higher orders are included. The Dirac operator for a product space such as that of the standard model, must now be taken to be of the form

$$D = D_1 \otimes \gamma_F + 1 \otimes D_F$$

instead of

$$D = D_1 \otimes 1 + \gamma_5 \otimes D_F$$

because γ_5 does not anticommute with D_1 on ∂M.

5.1 Dilaton as the Dynamical ScaleDilaton and Scale

Replacing the cutoff scale Λ in the spectral action, replacing $f(\frac{D^2}{\Lambda^2})$ by $f(P)$ where $P = e^{-\phi} D^2 e^{-\phi}$ modifies the spectral action with dilaton dependence to the form

$$\text{Tr } F(P) \simeq \sum_{n=0}^{6} f_{4-n} \int d^4x \sqrt{g} e^{(4-n)\phi} a_n \left(x, D^2\right)$$

One can then show that the dilaton dependence almost disappears from the action if one rescales the fields according to

$$G_{\mu\nu} = e^{2\phi} g_{\mu\nu}$$

$$H' = e^{-\phi} H$$

$$\psi' = e^{-\frac{3}{2}\phi} \psi$$

With this rescaling one finds the result that the spectral action is

$$
\begin{aligned}
& I\left(g_{\mu\nu} \rightarrow G_{\mu\nu}, H \rightarrow H', \psi \rightarrow \psi'\right) \\
& + \frac{24 f_2}{\pi 2} \int d^4 x \sqrt{G} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi
\end{aligned}
$$

scale invariant (independent of the dilaton field) except for the kinetic energy of the dilaton field ϕ. The dilaton field has no potential at the classical level. It acquires a Coleman-Weinberg potential through quantum corrections, and thus a vev. The dilaton acquires a very small mass. The Higgs sector in this case becomes identical with the Randall-Sundrum model. In that model there are two branes in a five dimensional space, one located at $x_5 = 0$ representing the invisible sector, and another located at $x_5 = \pi r_c$, the visible sector. The physical masses are set by the symmetry breaking scale $v = v_0 e^{-k r_c \pi}$ so that $m = m_0 e^{-k r_c \pi}$. If the bare symmetry breaking scale is taken at $m_0 \sim 10^{19}$ Gev, then by taking $k r_c \pi = 10$ one gets the low-energy mass scale $m \sim 10^2$ Gev. It is not surprising that the Randall-Sundrum scenario is naturally incorporated in the noncommutative geometric model, because intuitively one can think of the discrete space as providing the different brane sectors.

6 Parity violating terms

It is possible to add to the spectral action terms that will violate parity such as the gravitational term $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}$ and the non-abelian θ term $\epsilon^{\mu\nu\rho\sigma}V_{\mu\nu}^{m}V_{\rho\sigma}^{m}$. These arise by allowing for the spectral action to include the term

$$\mathrm{Tr}\left(\gamma G\left(\frac{D^2}{\Lambda^2}\right)\right)$$

where G is a function not necessarily equal to the function F, and

$$\gamma=\gamma_5\otimes\gamma_F$$

is the total grading. In this case it is easy to see that there are no contributions coming from a_0 and a_2 and the first new term occurs in a_4 where there are only two contributions:

$$\frac{1}{16\pi^2}\frac{1}{12}\mathrm{Tr}\left(\gamma_5\gamma_F\,\Omega_{\mu\nu}^2\right)=\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}(24-24)=0$$

and

$$-\frac{4}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(\left(\left(1-\left(\frac{1}{2}\right)^2(2)+\left(\frac{2}{3}\right)^2(3)+\left(\frac{1}{3}\right)^2(3)-\left(\frac{1}{6}\right)^2(3)\,(2)\right)3g_1^2B_{\mu\nu}B_{\rho\sigma}\right.\right.$$
$$\left.\left(-\left(\frac{1}{2}\right)^2(2)-\left(\frac{1}{2}\right)^2(2)\,(3)\right)3g_2^2W_{\mu\nu}^{\alpha}W_{\rho\sigma}^{\alpha}+\left(\left(\frac{1}{2}\right)^2(2)\,(1+1-2)\right)3V_{\mu\nu}^{m}V_{\rho\sigma}^{m}\right)$$
$$=-\frac{3}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(2g_1^2B_{\mu\nu}B_{\rho\sigma}-2g_2^2W_{\mu\nu}^{\alpha}W_{\rho\sigma}^{\alpha}\right)$$

Thus the additional terms to the spectral action, up to orders $\frac{1}{\Lambda^2}$, are

$$\frac{3G_0}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(2g_1^2B_{\mu\nu}B_{\rho\sigma}-2g_2^2W_{\mu\nu}^{\alpha}W_{\rho\sigma}^{\alpha}\right)$$

where $G_0=G\left(0\right)$. The $B_{\mu\nu}B_{\rho\sigma}$ is a surface term, while $W_{\mu\nu}^{\alpha}W_{\rho\sigma}^{\alpha}$ is topological, and both violate PC invariance. The surprising thing is the vanishing of both the gravitational PC violating term $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu ab}R_{\rho\sigma}{}^{ab}$ and the θ QCD term $\epsilon^{\mu\nu\rho\sigma}V_{\mu\nu}^{m}V_{\rho\sigma}^{m}$. In this way the θ parameter is naturally zero, and can only be generated by the higher order interactions. The reason behind the vanishing of both terms is that in these two sectors there is a left-right symmetry graded with the matrix γ_F giving an exact cancelation between the left-handed sectors and the right-handed ones. In other words the trace of γ_F vanishes and this implies that the index of the full Dirac operator, using the total grading, vanishes. There is one more condition to solve the strong CP problem which is to have the following condition on the mass matrices of the up quark and down quark

$$\det k^u\det k^d=\text{real}\,.$$

At present, it is not clear what condition must be imposed on the quarks Dirac operator, in order to obtain such relation. If this condition can be imposed naturally, then it will be possible to show that

$$\theta_{QT} + \theta_{QCD} = 0$$

at the tree level, and loop corrections can only change this by orders of less than 10^{-9}.

7 Order one and Beyond the SM

It appears that from experimental results that at present there are no indications of any new physics beyond the SM, but this does not rule out that some new physics will appear at very high energies. Indications that this is the case can be seen by the fact that the three gauge couplings do not meet at high energies as required by the spectral action. In addition the presence of the sigma field at energies of the order of 10^{11} Gev suggests that new physics would start to play a role at such high energies. Accepting this lead us to consider relaxing the order one condition and to investigate which model one gets.

The first order condition is what restricted a more general gauge symmetry based on the algebra $\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$ to the subalgebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

It is thus essential to understand the physical significance of such a requirement. In what follows we shall examine the more general algebra allowed without the first order condition, and shall show that the number of fundamental fermions is still dictated to be 16. We determine the inner automorphisms of the algebra $\mathcal{A}$ and show that the resulting gauge symmetry is a Pati-Salam type left-right model

$$SU(2)_R \times SU(2)_L \times SU(4)$$

where $SU(4)$ is the color group with the lepton number as the fourth color. In addition we observe that the Higgs fields appearing in $A_{(2)}$ are composite and depend quadratically on those appearing in $A_{(1)}$ provided that the initial Dirac operator (without fluctuations) satisfy the order one condition. Otherwise, there will be additional fundamental Higgs fields. In particular, the representations of the fundamental Higgs fields when the initial Dirac operator satisfies the order one condition are $(2_R, 2_L, 1)$, $(2_R, 1_L, 4)$ and $(1_R, 1_L, 1+15)$ with respect to $SU(2)_R \times SU(2)_L \times SU(4)$. When the order one condition is not satisfied for the initial Dirac operator, the representations of the additional Higgs fields are $(3_R, 1_L, 10)$, $(1_R, 1_L, 6)$ and $(2_R, 2_L, 1+15)$. There are simplifications if the Yukawa coupling of the up quark is equated with that of the neutrino and of the down quark equated with that of the electron. In addition the $1+15$ of $SU(4)$ decouple if we assume that at unification scale there is exact $SU(4)$ symmetry between the quarks and leptons. The resulting model is very similar to the one considered by Marshak and Mohapatra.

When one considers inner fluctuations of the Dirac operator one finds that the gauge transformation takes the form

$$D_A \to U D_A U^*, \qquad U = u\, Ju\, J^{-1}, \qquad u \in \mathcal{U}(\mathcal{A})$$

which implies that

$$A \to u\, A u^* + u\delta(u^*) .$$

This in turn gives

$$A_{(1)} \to u A_{(1)} u^* + u\,[D, u^*] \in \Omega^1_D(\mathcal{A})$$

$$A_{(2)} \to Ju\, J^{-1} A_{(2)} Ju^*\, J^{-1} + Ju\, J^{-1} \left[u\,[D, u^*]\,, Ju^{\,*} J^{-1} \right]$$

where the $A_{(2)}$ in the right hand side is computed using the gauge transformed $A_{(1)}$. Thus $A_{(1)}$ is a one-form and behaves like the usual gauge transformations. On the other hand $A_{(2)}$ transforms non-linearly and includes terms with quadratic dependence on the gauge transformations.

We now proceed to compute the Dirac operator on the product space $M \times F$. The initial operator is given by

$$D = \gamma^\mu D_\mu \otimes 1 + \gamma_5 D_F$$

where $\gamma^\mu D_\mu = \gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega_\mu^{\ \ ab}\gamma_{ab}\right)$ is the Dirac operator on the four dimensional spin manifold. Then the Dirac operator including inner fluctuations is given by

$$D_A = D + A_{(1)} + JA_{(1)}J^{-1} + A_{(2)}$$

$$A_{(1)} = \sum a\left[D, b\right]$$
$$A_{(2)} = \sum a\left[JA_{(1)}J^{-1}, b\right].$$

The computation is very involved thus for clarity we shall collect all the details in the appendix and only quote the results in what follows. The different components of the operator D_A are then given by

$$(D_A)_{\dot{a}I}^{\dot{b}J} = \gamma^\mu \left(D_\mu \delta_{\dot{a}}^{\dot{b}}\delta_I^J - \frac{i}{2}g_R W_{\mu R}^\alpha \left(\sigma^\alpha\right)_{\dot{a}}^{\dot{b}}\delta_I^J - \delta_{\dot{a}}^{\dot{b}}\left(\frac{i}{2}gV_\mu^m\left(\lambda^m\right)_I^{\ J} + \frac{i}{2}gV_\mu\delta_I^J\right)\right)$$
$$(D_A)_{aI}^{bJ} = \gamma^\mu \left(D_\mu \delta_a^b\delta_I^J - \frac{i}{2}g_L W_{\mu L}^\alpha \left(\sigma^\alpha\right)_a^b\delta_I^J - \delta_a^b\left(\frac{i}{2}gV_\mu^m\left(\lambda^m\right)_I^{\ J} + \frac{i}{2}gV_\mu\delta_I^J\right)\right)$$

where the fifteen 4×4 matrices $\left(\lambda^m\right)_I^{\ J}$ are traceless and generate the group $SU(4)$ and $W_{\mu R}^\alpha$, $W_{\mu L}^\alpha$, V_μ^m are the gauge fields of $SU(2)_R$, $SU(2)_L$, and $SU(4)$. The requirement that A is unimodular implies that

$$\mathrm{Tr}\left(A\right) = 0$$

which gives the condition

$$V_\mu = 0.$$

In addition we have

$$(D_A)_{\dot{a}I}^{bJ} = \gamma_5 \left(\left(k^\nu \phi_{\dot{a}}^b + k^e \widetilde{\phi}_{\dot{a}}^b \right) \Sigma_I^J + \left(k^u \phi_{\dot{a}}^b + k^d \widetilde{\phi}_{\dot{a}}^b \right) \left(\delta_I^J - \Sigma_I^J \right) \right) \equiv \gamma_5 \Sigma_{\dot{a}I}^{bJ} \tag{4}$$

$$(D_A)_{\dot{a}I}^{\dot{b}' J'} = \gamma_5 k^{*\nu_R} \Delta_{\dot{a}J} \Delta_{\dot{b}I} \equiv \gamma_5 H_{\dot{a}I\dot{b}J}$$

where the Higgs field $\phi_{\dot{a}}^b$ is in the $\left(2_R, \overline{2}_L, 1\right)$ of the product gauge group $SU(2)_R \times SU(2)_L \times SU(4)$, and $\Delta_{\dot{a}J}$ is in the $(2_{R,}, 1_L, 4)$ representation while Σ_I^J is in the $(1_R, 1_L, 1+15)$ representation. The field $\widetilde{\phi}_{\dot{a}}^b$ is not an independent field and is given by

$$\widetilde{\phi}_{\dot{a}}^b = \tau_2 \overline{\phi}_{\dot{a}}^b \tau_2.$$

Note that the field Σ_I^J decouples (and set to $\delta_I^1 \delta_1^J$) in the special case when there is lepton and quark unification of the couplings

$$k^\nu = k^u, \qquad k^e = k^d.$$

In case when the initial Dirac operator satisfies the order one condition, then the $A_{(2)}$ part of the connection becomes a composite Higgs field where the Higgs field $\Sigma_{\dot{a}I}^{bJ}$ is formed out of the products of the fields $\phi_{\dot{a}}^b$ and Σ_I^J while the Higgs field $H_{\dot{a}I\dot{b}J}$ is made from the product of $\Delta_{\dot{a}J}\Delta_{\dot{b}I}$. For generic initial Dirac operators, the field $\left(A_{(2)}\right)_{\dot{a}I}^{bJ}$ becomes independent. The fields

$\Sigma_{\dot{a}I}^{bJ}$ and $H_{\dot{a}I\dot{b}J}$ will then not be defined through equation 4 and will be in the $(2_R, 2_L, 1+15)$ and $(3_R, 1_L, 10) + (1_R, 1_L, 6)$ representations of $SU(2)_R \times SU(2)_L \times SU(4)$. In addition, for generic Dirac operator one also generate the fundamental field $(1, 2_L, 4)$. The fact that inner automorphisms form a semigroup implies that the cases where the Higgs fields contained in the connections $A_{(2)}$ are either independent fields or depend quadratically on the fundamental Higgs fields are disconnected.

The interesting question that needs to be addressed is whether the structure of the connection is preserved at the quantum level. This investigation must be performed in such a way as to take into account the noncommutative structure of the space. At any rate, we have here a clear advantage over grand unified theories which suffers of having arbitrary and complicated Higgs representations . In the noncommutative geometric setting, this problem is now solved by having minimal representations of the Higgs fields. Remarkably, we note that a very close model to the one deduced here is the one considered by Marshak and Mohapatra where the $U(1)$ of the left-right model is identified with the $B-L$ symmetry. They proposed the same Higgs fields that would result starting with a generic initial Dirac operator not satisfying the first order condition. Although the broken generators of the $SU(4)$ gauge fields can mediate lepto-quark interactions leading to proton decay, it was shown that in all such types of models with partial unification, the proton is stable. In addition this type of model arises in the first phase of breaking of $SO(10)$ to $SU(2)_R \times SU(2)_L \times SU(4)$ and these have been extensively studied. The recent work in considers noncommutative grand unification based on the $k = 8$ algebra $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ keeping the first order condition.

7.1 The Spectral Action for the $SU(2)_R \times SU(2)_L \times SU(4)$ model

The bosonic action is given by

$$\text{Trace}\,(f(D_A/\Lambda))$$

which gives

$$
\begin{aligned}
S_{\mathrm{b}} &= \frac{24}{\pi^2}F_4\Lambda^4\int d^4x\sqrt{g} \\
&- \frac{2}{\pi^2}F_2\Lambda^2\int d^4x\sqrt{g}\left(R+\frac{1}{4}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right) \\
&+ \frac{1}{2\pi^2}F_0\int d^4x\sqrt{g}\left[\frac{1}{30}\left(-18C_{\mu\nu\rho\sigma}^2+11R^*R^*\right)+g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2+g_R^2\left(W_{\mu\nu R}^{\alpha}\right)^2+g^2\left(V_{\mu\nu}^{m}\right)^2\right. \\
&+ \nabla_\mu\Sigma_{aI}^{\dot{c}K}\nabla^\mu\Sigma_{\dot{c}K}^{aI}+\frac{1}{2}\nabla_\mu H_{\dot{a}I\dot{b}J}\nabla^\mu H^{\dot{a}I\dot{b}J}+\frac{1}{12}R\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right) \\
&\left.+\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2+2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ}+\Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\right].
\end{aligned}
$$

The physical content of this action is a cosmological constant term, the Einstein Hilbert term R, a Weyl tensor square term $C_{\mu\nu\rho\sigma}^2$, kinetic terms for the $SU(2)_R\times SU(2)_L\times SU(4)$ gauge fields, kinetic terms for the composite Higgs fields $H_{\dot{a}I\dot{b}J}$ and $\Sigma_{bJ}^{\dot{c}K}$ as well as mass terms and quartic terms for the Higgs fields. This is a grand unified Pati-Salam type model with a completely fixed Higgs structure which we expect to spontaneously break at very high energies to the $U(1)\times SU(2)\times SU(3)$ symmetry of the SM. We also notice that this action gives the gauge coupling unification

$$g_R = g_L = g.$$

A test of this model is to check whether this relation when run using RG equations would give values consistent with the values of the gauge couplings for electromagnetic, weak and strong interactions at the scale of the Z -boson mass. Having determined the full Dirac operators, including fluctuations, we can write all the fermionic interactions including the ones with the gauge vectors and Higgs scalars. It is given by

$$\int d^4x\sqrt{g}\left\{\psi^*_{\dot{a}I}\gamma^\mu\left(D_\mu\delta^{\dot{b}}_{\dot{a}}\delta^J_I-\frac{i}{2}g_R W^\alpha_{\mu R}\left(\sigma^\alpha\right)^{\dot{b}}_{\dot{a}}\delta^J_I-\delta^{\dot{b}}_{\dot{a}}\left(\frac{i}{2}gV^m_\mu\left(\lambda^m\right)^J_I+\frac{i}{2}gV_\mu\delta^J_I\right)\right)\psi_{\dot{b}J}\right.$$
$$+\psi^*_{aI}\gamma^\mu\left(D_\mu\delta^b_a\delta^J_I-\frac{i}{2}g_L W^\alpha_{\mu L}\left(\sigma^\alpha\right)^b_a\delta^J_I-\delta^b_a\left(\frac{i}{2}gV^m_\mu\left(\lambda^m\right)^J_I+\frac{i}{2}gV_\mu\delta^J_I\right)\right)\psi_{bJ}$$
$$\left.+\psi^*_{\dot{a}I}\gamma_5\Sigma^{bJ}_{\dot{a}I}\psi_{bJ}+\psi^*_{aI}\gamma_5\Sigma^{\dot{b}J}_{aI}\psi_{\dot{b}J}+C\psi_{\dot{a}I}\gamma_5H^{\dot{a}I\dot{b}J}\psi_{\dot{b}J}+\text{h.c}\right\}$$

It is easy to see that this model truncates to the Standard Model. The Higgs field $\phi^b_{\dot{a}}=(2_R,2_L,1)$ must be truncated to the Higgs doublet H by writing

$$\phi^b_{\dot{a}}=\delta^{\dot{1}}_{\dot{a}}\epsilon^{bc}H_c.$$

The other Higgs field $\Delta_{\dot{a}I}=(2_R,1,4)$ is truncated to a real singlet scalar field

$$\Delta_{\dot{a}I}=\delta^{\dot{1}}_{\dot{a}}\delta^1_I\sqrt{\sigma}.$$

Needless to say that it is difficult to determine all allowed vacua of this potential, especially since there is dependence of order eight on the fields. It is possible, however, to expand this potential around the vacuum that we started with which breaks the gauge symmetry directly from $SU(2)_R \times SU(2)_L \times SU(4)$ to $U(1)_{\text{em}} \times SU(3)_{\text{c}}$. Explicitly, this vacuum is given by

$$\left\langle \phi_{\dot{a}}^{b} \right\rangle = v\delta_{\dot{a}}^{\dot{1}}\delta_{1}^{b} \qquad \left\langle \Sigma_{J}^{I} \right\rangle = u\delta_{1}^{I}\delta_{J}^{1} \qquad \left\langle \Delta_{\dot{a}J} \right\rangle = w\delta_{\dot{a}}^{\dot{1}}\delta_{J}^{1}.$$

Relaxing the order one condition which may be required in the process of renormalizing the spectral action leads uniquely to the Pati-Salam model with $SU(2)_R \times SU(2)_L \times SU(4)$ symmetry unifying leptons and quarks with the lepton number as the fourth color. The Higgs fields are fixed and belong to the 16×16 and $16 \times \overline{16}$ products with respect to the Pati-Salam group. Because of the semi-group structure of the inner fluctuations the Higgs fields may all be independent of each other, or the $A_{(2)}$ part of the connection depending on the $A_{(1)}$ parts provided that the initial Dirac operator is taking to satisfy the order one condition with respect to the SM algebra. The model, unlike other unification models does not suffer from prton decay and is not ruled out experimentally. A lot of work remains to be done to investigate this model and study all its possible breakings from the high energy to low energies. Of interest is to determine whether the additional fields present will modify the running of the gauge couplings allowing for the meetings of these couplings at very high energies.

8 Conclusions and outlook

Noncommutative geometry methods are very effective in understanding and predicting the nature of space-time and have come a long way. It is now important to push this success further by addressing problems such as the quantization of gravity using these tools.