On the Relation Betweeen Gauge and Phase Symmetries

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ERC Project *Philosophy of Canonical Quantum Gravity*

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Symmetries & Reduction

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$\Downarrow$

Reduction

... in the amount of (invariant) information...

... that is necessary to completely describe a system.

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$\Downarrow$

Reduction

... in the amount of (invariant) information...

... that is necessary to completely describe a system.

- Example: ***gauge theories*** (or ***constrained Hamiltonian systems***):

2n degrees of freedom + k first-class constraints

$\Downarrow$

2(n -k) *physical* degrees of freedom

From Classical to Quantum

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- The transition from classical to quantum mechanics...

  ... entails a reduction in the number of obs. that are necessary to define a physical state:

2n *classical* observables q *and* p

$$\Downarrow$$

n *quantum* observables q *or* p.

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- In the simplest case, the *phase invariance* of $|p_i\rangle$ under translations in q

$$q_0 \cdot |p_i\rangle \mapsto e^{2\pi i q_0 p_i} |p_i\rangle \approx |p_i\rangle$$

  can be interpreted by saying that the position q of $|p_i\rangle$ is completely “undetermined”.

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- **Heisenberg indeterminacy principle** generalizes this reduction to more gral. states (e.g. coherent states).

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- In analogy to gauge theories, we could try to understand this transition...

  ... as a reduction induced by some form of symmetry.

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- In analogy to gauge theories, we could try to understand this transition...

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- Far from being a mere analogy, I will argue that

  ... the ***quantum phase symmetries*** can be understood...

  ... as a consequence of the same formalism underlying the gauge symmetries, i.e. the

  symplectic reduction procedure.

Hamiltonian G-manifolds

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- Let (M, ω, μ) be a ***Hamiltonian G-manifold***, i.e. a connected symplectic manifold endowed

.with an action $\Phi : G \times M \to M$ of a Lie group G preserving ω (i.e. $\Phi_g^* \omega = \omega$ for all $g \in G$).

.with an ***equivariant moment map*** (introduced by **J.-M. Souriau**)

$$\mu : M \to \mathfrak{g}^*$$

i.e. a (Poisson) map intertwining the G-action on M and the G-co-adjoint action on $\mathfrak{g}^*$:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle Ad^*_{g^{-1}}} \\ M & \xrightarrow[\mu]{} & \mathfrak{g}^*. \end{array}$$

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Moment Map

- Given $X_i \in \mathfrak{g}$, the ***moment map***

$$\mu : M \rightarrow \mathfrak{g}^*$$

defines a ***generating function of the group action*** on M

$$f_i(m) = \langle \mu(m), X_i \rangle, \qquad X_i \in \mathfrak{g}$$

such that its ***symplectic gradient***

$$v_{f_i} = \omega^{-1} df_i$$

is the ***fundamental vector field*** that infinitesimally generates the G-action on M.

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- The fact of considering M *over* $\mathfrak{g}^*$ implies that there is a privileged family $\{f_i\}_{X_i \in \mathfrak{g}}$ of observables on M (i.e. the ***generating functions*** f_i).

What is $\mathfrak{g}^*$ useful for?

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- A Hamiltonian G-manifold (M, ω, μ) is not only endowed with a symplectic G-action, but also with a map towards $\mathfrak{g}^*$.

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- A Hamiltonian G-manifold (M, ω, μ) is not only endowed with a symplectic G-action, but also with a map towards $\mathfrak{g}^*$.

- Now, how can we interpret this relation between M and $\mathfrak{g}^*$?

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- Roughly speaking, $\mathfrak{g}^*$ encodes the **unitary representation theory of G**.

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- Roughly speaking, $\mathfrak{g}^*$ encodes the **unitary representation theory of G**.
- Firstly, $\mathfrak{g}^*$ **is a Poisson manifold** with respect to the so-called **Lie-Poisson structure**

$$\{f, g\}(x) = \langle x, [df(x), dg(x)]\rangle \qquad f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$$

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- Secondly,

Symplective leaves of $\mathfrak{g}^* =$ Coadjoint orbits $\mathcal{O}$ of $G \circlearrowleft \mathfrak{g}^*$

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- Secondly,

Symplective leaves of $\mathfrak{g}^*$ = Coadjoint orbits $\mathcal{O}$ of $G \circlearrowleft \mathfrak{g}^*$

- The coadjoint orbits $\mathcal{O}$ are endowed with a canonical G-invariant symplectic structure $\omega_{\mathcal{O}}$.

Kirillov's Orbit Method

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- For certain G, Kirillov's ***orbit method*** establishes a correspondence

$$\mathfrak{g}^*_{\mathbb{Z}}/G \sim \hat{G},$$

(where $\hat{G}$ is the ***unitary dual*** of G) given by

$$\mathcal{O} \rightsquigarrow \mathcal{H}_{\mathcal{O}},$$

where $\mathcal{H}_{\mathcal{O}}$ is the Hilbert space obtained by applying the geometric quantization procedure to the symplectic manifold $\mathcal{O}$...

or by applying the functor Ind^G_H to the 1-dim unirrep ρ^H_ξ of $H = exp(\mathfrak{h})$ defined by $\xi \in \mathcal{O}$ where $\mathfrak{h} \subset \mathfrak{g}$ is a max. subalg. subordinated to ξ:

$$\langle \xi, [\mathfrak{h}, \mathfrak{h}] \rangle = 0.$$

G-Homogeneous Symplectic Manifolds in $\mathfrak{g}^*$ $\rightsquigarrow$ Irreducible Unitary Representations of G

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G-Homogeneous Symplectic Manifolds in $\mathfrak{g}^*$ $\rightsquigarrow$ Irreducible Unitary Representations of G

- If G is abelian, each $\xi \in \mathfrak{g}^*_{\mathbb{Z}}$ is a coadjoint orbit defining a 1-dim. unirrep of G:

$$\begin{aligned} \rho_\xi : G &\rightarrow U(1) \\ e^X &\mapsto e^{2\pi i \langle \xi, X \rangle}, \qquad X \in \mathfrak{g}. \end{aligned}$$

Kirillov's Conjecture

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- Exactly as the G-action on the *homogeneous* symplectic orbit $\mathcal{O} \subset \mathfrak{g}^*$ is lifted to an *irreducible* unitary action on $\mathcal{H}_{\mathcal{O}}$...

  ... we could expect the G-action on M to be lifted to a unitary action on $\mathcal{H}_M$.

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  ... we could expect the G-action on M to be lifted to a unitary action on $\mathcal{H}_M$.

- Since M is not in gral. G-homogeneous, the lifted unitary action will not in gral. be irreducible:

$$\mathcal{H}_M = \bigoplus_{\mathcal{O} \subset \mathfrak{g}^*} m(\mathcal{O}, M)\mathcal{H}_{\mathcal{O}},$$

where $m(\mathcal{O}, M) \doteq dim(Hom_G(\mathcal{H}_{\mathcal{O}}, \mathcal{H}_M))$ is the multiplicity with which the unirrep $\mathcal{H}_{\mathcal{O}}$ occurs in $\mathcal{H}_M$.

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- **Kirillov's conjecture**: μ tells which unirreps of G occur in $\mathcal{H}_M$.

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- **Kirillov's conjecture**: μ tells which unirreps of G occur in $\mathcal{H}_M$.

- **Guillemin-Sternberg conjecture**: μ also gives $m(\mathcal{O}, M)$.

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- **Kirillov's conjecture**: μ tells which unirreps of G occur in $\mathcal{H}_M$.

- **Guillemin-Sternberg conjecture**: μ also gives $m(\mathcal{O}, M)$.

- Hence, μ encodes the ***quantization of M over $\mathfrak{g}^*$***, i.e. the quantization of M with respect to the observable algebra induced by the G-action on M.

ξ-Symplectic Quotients

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- We must learn how to use μ for "pulling-back" the G-unirreps supported by $\mathfrak{g}^*$ to M. Let's consider the case of an abelian G...

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- Since the unirreps $\mathcal{H}_\xi$ ($\xi \in \mathfrak{g}^*$) are "supported" by ξ, let's consider the states in M corresponding to a fixed "value" ξ of the "momentum", that is

$$\mu^{-1}(\xi).$$

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- (Shifted) **Mardsen-Weinstein reduction theorem**:

$$M_\xi \doteq \mu^{-1}(\xi)/G$$

is a symp. manifold called the ***ξ-symplectic quotient of M***.

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- So, M_ξ is the symp. counterpart of ξ in M.

A Restriction entails a Projection

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- In gauge-theoretic terms, when we fix the "value" of the "momentum" μ to ξ by means of the ***restriction*** to the "ξ-constraint surface"

$$\mu^{-1}(\xi) \subset M,$$

... the "conjugate coordinate" acted upon by G becomes completely "*undetermined*"...

.... in the sense that it is "*gauged out*" by means of the ***projection***

$$\mu^{-1}(\xi) \twoheadrightarrow M_{\xi}.$$

How should we interpret M_ξ?

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- We shall argue...

1) that the ξ-symplectic quotient

$$M_\xi \doteq \mu^{-1}(\xi)/G$$

is the "***moduli space***" parameterizing the category-theoretical ***symplectic ξ-points*** of M.

2) that the notion of symplectic point elicits a category-theoretical interpretation of Heisenberg indeterminacy principle.

Weinstein’s Symplectic Creed

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“The Heisenberg uncertainty principle says that it is impossible to determine simultaneously the position and momentum of a quantum-mechanical particle. This can be rephrased as follows: the smallest subsets of classical phase space in which the presence of a quantum-mechanical particle can be detected are its Lagrangian submanifolds. For this reason it makes sense to regard the Lagrangian submanifolds of phase space as being its true “points”.”

V. Guillemin and S. Sternberg, *Geometric Quantization and Multiplicities of Group Representations*, 1982.

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- This notion of ***Lagrangian true "points"*** acquires a precise category-theoretical meaning in the framework of Weinstein's symplectic "category".

Category-Theoretical “Points”

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- A point x in a manifold M can be identified with the morphism

$$\varphi_x : \{*\} \to M$$

given by

$$\{*\} \mapsto x$$

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- A point x in a manifold M can be identified with the morphism

$$\varphi_x : \{*\} \to M$$

given by

$$\{*\} \mapsto x$$

- More generally, given two objects A and B in a category, the morphisms

$$B \to A$$

define the so-called ***B-points of A***.

Weinstein's Symplectic "Category"

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- **Objects**:

  .Symplectic manifolds (M, ω).

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- **Objects**:

  .Symplectic manifolds (M, ω).

- **Morphisms** (or **Lagrangian correspondences**) $(M_2, \omega_2) \to (M_1, \omega_1)$:

$$Hom_{Symp}(M_2, M_1) = \left\{ L_{2,1} \hookrightarrow M_1 \times M_2^- \right\}$$

where $(M_2^-, -\omega_2)$ is the *dual* of (M_2, ω_2) and

$$(M_1 \times M_2^-, \pi_1^* \omega_1 - \pi_2^* \omega_2),$$

is the product symplectic manifold.

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- In particular, a symplectomorphism defines a Lagrangian corresp.

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  is the product symplectic manifold.

- In particular, a symplectomorphism defines a Lagrangian corresp.

- The ***symplectic points*** of (M, ω) is given by the morphisms in

$$Hom_{Symp}((*, 0), (M, \omega)) = \{L \hookrightarrow M \times \{*\} \simeq M\},$$

  i.e. by the Lagrangian submanifolds of M.

Weinstein's G-Symplectic "Category"

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- **Objects**:

  .Hamiltonian G-manifolds (M, ω, μ).

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$$
\begin{array}{ccccc}
L_{2,1} & \hookrightarrow & M_1 \times_{\mathfrak{g}^*} M_2^- & \longrightarrow & M_2^- \\
 & & \downarrow & & \downarrow \mu_2 \\
 & & M_1 & \xrightarrow{\mu_1} & \mathfrak{g}^*
\end{array}
$$

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- In other terms,

$$Hom_{G\text{-}Symp}(M_2, M_1) = \left\{ L_{2,1} \hookrightarrow \Phi^{-1}(0) \subset M_1 \times M_2^- \right\},$$

  where

$$(M_1 \times M_2^-, \pi_1^* \omega_1 - \pi_2^* \omega_2, \Phi \doteq \mu_1 - \mu_2),$$

  is the product Hamiltonian G-manifold.

Classical Intertwiner Spaces

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- It can be shown (♣) that $L_{2,1} \subset M_1 \times_{\mathfrak{g}^*} M_2^-$ are G-invariant...

  ... and that there is a bijection

$$Hom_{G\text{-}Symp}(M_2, M_1) \simeq \left\{ L \subset (M_1 \times_{\mathfrak{g}^*} M_2^-)/G \right\}.$$

♣ Xu, P. [1994]: "Classical Intertwiner Space and Quantization," Commun. Math. Phys. 164, 473-488.

Classical Intertwiner Spaces

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- Under nice conditions, $(M_1 \times_{\mathfrak{g}^*} M_2^-)/G$ is a symplectic manifold...

  ... whose symplectic points are the classical intertwiners over $\mathfrak{g}^*$ between M_2 and M_1...

  ... or, in category-theoretical terms, the M_2-sympletic points of M_1.

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ξ-Symplectic Points of M

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- In particular, the morphisms $(\xi, 0, \mu_\xi : \xi \mapsto \xi) \to (M, \omega, \mu)$ are given by the Lagrang. subman. of

$$(M \times_{\mathfrak{g}^*} \xi^-)/G = \Phi^{-1}(0)/G,$$

where the twisted moment map is

$$\begin{aligned} \Phi : M \times \xi^- &\to \mathfrak{g}^* \\ (m, \xi) &\mapsto \mu(m) - \xi. \end{aligned}$$

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- Now, since $\Phi^{-1}(0) \simeq \mu^{-1}(\xi) \subset M$, then $\Phi^{-1}(0)/G \simeq M_\xi$.

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- All in all,

$$Hom_{G\text{-}Symp}(\xi, M) = \{L \subset M_\xi\}$$

i.e. the symplectic points of M_ξ are in correspondence with the ξ-points of M.

M_ξ can be interpreted as the *moduli space* of symplectic ξ-points of M.

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Quantum Intertwiner Spaces

- Guillemin and Sternberg (1982) showed (for particular M and G) that the (geometric) quantization of the ***classical intertwiner space:***

$$M_{\mathcal{O}} \cong Hom_{G\text{-}Symp}(\mathcal{O}, M)$$

between a coadjoint orbit $\mathcal{O}$ and M yields the ***quantum intertwiner space:***

$$\mathcal{H}_{M_{\mathcal{O}}} \cong Hom_G(\mathcal{H}_{\mathcal{O}}.\mathcal{H}_M),$$

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$$\mathcal{H}_{M_{\mathcal{O}}} \cong Hom_G(\mathcal{H}_{\mathcal{O}}.\mathcal{H}_M),$$

- Whereas $M_{\mathcal{O}}$ parameterizes the **symplectic G-morphisms**

$$\mathcal{O} \rightarrow M,$$

$\mathcal{H}_{M_{\mathcal{O}}}$ parameterizes the **unitary G-intertwiners**

$$\mathcal{H}_{\mathcal{O}} \rightarrow \mathcal{H}_M$$

Quantum Intertwiner Spaces

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$$\mathcal{H}_{\mathcal{O}} \to \mathcal{H}_M$$

- ***Kirillov's conjecture revisited***: The unirrep $\mathcal{H}_{\mathcal{O}}$ occurs in $\mathcal{H}_M$ if M has symplectic $\mathcal{O}$-points where the multiplicity is given by

$$m(\mathcal{O}, M) = dim(\mathcal{H}_{M_{\mathcal{O}}}).$$

Mardsen-Weinstein 0-Reduction

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- A ***gauge theory*** is a Ham. G-manifold (M, ω, μ) such that the Ham. equations constraint the solutions to be in the ***constraint surface***

$$\Sigma = \mu^{-1}(0) \subset M.$$

Mardsen-Weinstein 0-Reduction

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$$\Sigma = \mu^{-1}(0) \subset M.$$

- The Mardsen-Weinstein reduction theorem shows that the 0-symplectic quotient

$$M_0 \simeq \mu^{-1}(0)/G$$

has a canonical symplectic form ω_{M_0} satisfying

$$\pi^* \omega_{M_0} = \iota^* \omega_M ,$$

where

$$\begin{array}{ccc} \mu_M^{-1}(0) & \overset{\iota}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_0 \doteq \mu^{-1}(0)/G. & & \end{array}$$

Quantization Commutes with 0-Reduction

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- In this case, the Guillemin-Sternberg conjecture is

$$\mathcal{H}_{M_0} \cong Hom_G(\mathcal{H}_0, \mathcal{H}_M),$$

or, in other terms,

$$\mathcal{H}_{M_0} \simeq \mathcal{H}_M^G,$$

where $\mathcal{H}_M^G$ is the space of G-invariant states in $\mathcal{H}_M$.

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- Diagramatically,

$$\begin{array}{ccc} M & \overset{\text{Quantization}}{\rightsquigarrow} & \mathcal{H}_M \\ \text{0-symplectic reduction} \downarrow & & \downarrow G\text{-invariant states} \\ M_0 & \rightsquigarrow & \mathcal{H}_{M_0} \simeq \mathcal{H}_M^G, \end{array}$$

Quantum G-invariance $\leftrightsquigarrow$ Classical 0-symplectic reduction.

Gauge Theories

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- A ***gauge theory*** is a given by a Ham. G-manifold such that the restriction of the theory to the classical intertwiner space

$$M_0 \simeq Hom(0, M) \qquad\qquad 0 \in \mathfrak{g}^*$$

containing the 0-symplectic points of M...

... implies that the resulting quantum theory only includes G-invariant states.

Gauge Theories

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- In this case, G is called the ***gauge group*** of the theory and the generating functions of the G-action

$$G_i(m) = \langle \mu(m), X_i \rangle \qquad\qquad X_i \in \mathfrak{g}$$

are called ***constraints***.

Gauge Groups vs. Phase Groups

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- We are here interested in *ordinary* theories defined on a Ham. G-manifolds (M, ω, μ),

  ... where by *ordinary* we mean that ***the theories are* not *constrained to a unique value of*** μ.

Gauge Groups vs. Phase Groups

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- We are here interested in *ordinary* theories defined on a Ham. G-manifolds (M, ω, μ),

... where by *ordinary* we mean that ***the theories are* not *constrained to a unique value of*** μ.

- We shall call G the ***phase group*** and the non-constrained generating functions of the G-action

$$f_i(m) = \langle \mu(m), X_i \rangle$$

phase observables.

$\mathcal{O}$-Symplectic Reductions

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- Differently from the constraints G_a, the phase observables f_i do not select a single unirrep of G.

$\mathcal{O}$-Symplectic Reductions

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- Differently from the constraints G_a, the phase observables f_i do not select a single unirrep of G.
- Therefore, while a ***gauge group action*** defines a unique 0-symplectic quotient

$$M_0 \doteq \mu^{-1}(0)/G,$$

... associated to the trivial unirrep of G...

... a phase group action defines a different ***$\mathcal{O}$-symplectic quotient***

$$M_{\mathcal{O}} \doteq \mu^{-1}(\mathcal{O})/G$$

for each unirrep $\mathcal{O} \subset \mathfrak{g}^*_{\mathbb{Z}}$ of the phase group G.

$\mathcal{O}$-Symplectic Reductions

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$$M_{\mathcal{O}} \doteq \mu^{-1}(\mathcal{O})/G$$

for each unirrep $\mathcal{O} \subset \mathfrak{g}^*_{\mathbb{Z}}$ of the phase group G.

- The phase G-action on M entails the existence of a whole set of $\mathcal{O}$-symplectic quotients $M_{\mathcal{O}}$.

Quantization Commutes with ξ-Reduction

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- The Guillemin & Sternberg's conjecture for a ξ-symplectic quotient with G abelian states that this diagram commutes:

$$
\begin{array}{ccc}
M & \overset{\text{Quantization}}{\rightsquigarrow} & \mathcal{H}_M \\
\xi\text{-symplectic reduction} \; \wr\!\downarrow & & \wr\!\downarrow \; (G,\xi)\text{-phase invariant states} \\
M_\xi & \rightsquigarrow & \mathcal{H}_{M_\xi} \simeq \mathcal{H}_M^{(G,\xi)},
\end{array}
$$

.$\mathcal{H}_M^{(G,\xi)}$ is the space of (G,ξ)***-phase invariant states*** in $\mathcal{H}_M$,...

... i.e. the states that are invariant modulo a phase factor given by the 1-dim. unirrep ρ_ξ^G of G defined by ξ:

$$
\begin{aligned}
\rho_\xi^G : G \times \mathcal{H}_M^{(G,\xi)} &\rightarrow \mathcal{H}_M^{(G,\xi)} \\
(e^X, |\xi, \ldots\rangle) &\mapsto e^{2\pi i \langle \xi, X\rangle} |\xi, \ldots\rangle,
\end{aligned}
$$

Gauge Invariance vs. Phase Invariance

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- The (G, ξ)-phase invariance of quantum states is the quantum counterpart of the symplectic reduction with respect to a non-zero $\xi \in \mathfrak{g}^*$.

Quantum phase invariance is the generalization...

... of the strict gauge invariance...

.... to the case of ξ-symplectic reductions with $\xi \neq 0$.

From Symplectic to Phase Symmetries

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- We have argued that the existence of (G, ξ)-phase invariant states in $\mathcal{H}_M$...

  ... results from the existence of ξ-symplectic points in M.

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From Symplectic to Phase Symmetries

- We have argued that the existence of (G, ξ)-phase invariant states in $\mathcal{H}_M$...

  ... results from the existence of ξ-symplectic points in M.

- Far from being structureless set-theoretic points,...

  ... the ξ-symplectic points of M are non-trivial subman. of M endowed with a G-action.

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  ... results from the existence of ξ-symplectic points in M.

- Far from being structureless set-theoretic points,...

  ... the ξ-symplectic points of M are non-trivial subman. of M endowed with a G-action.

- The (G, ξ)-phase invariance of the states $|\xi, \ldots\rangle$,...

  ... i.e. the "indeterminacy" in the variable acted upon by G...

  ... is the quantum counterpart of the fact...

  ... that the corresponding ξ-symplectic points of M have an internal G-symmetry.

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For Instance...

- Let’s consider the simplest case of $G = \mathbb{R}$ acting on $M = T^*\mathbb{R}$ by

$$q_0 \cdot (q, p) \mapsto (q + q_0, p)$$

with moment map

$$\begin{aligned} \mu : T^*\mathbb{R} &\rightarrow \mathbb{R} \\ (q, p) &\mapsto p \end{aligned}$$

For Instance...

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$$\begin{array}{rcl} \mu : T^*\mathbb{R} & \rightarrow & \mathbb{R} \\ (q, p) & \mapsto & p \end{array}$$

- The p_i-symplectic quotient

$$M_{p_i} = \mu^{-1}(p_i)/G = \{*\}$$

contains the unique symplectic p_i-point of M.

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Phasing Out the Position

- According to

$$[\text{Quantization}, \xi\text{-Reduction}] = 0,$$

the (1-dimensional) quantization of M_{p_i} yields the (unique) (G, p_i)-phase invariant state $|p_i\rangle$ in $\mathcal{H}_M$:

$$q_0 \cdot |p_i\rangle \mapsto e^{2\pi i q_0 p_i} |p_i\rangle \approx |p_i\rangle.$$

Phasing Out the Position

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$$q_0 \cdot |p_i\rangle \mapsto e^{2\pi i q_0 p_i} |p_i\rangle \approx |p_i\rangle.$$

- The indeterminacy in the position q of the state $|p_i\rangle$ is a symptom of the fact...

... that the unique symplectic p_i-point of M has an internal symmetry under translations in q.

On Symplectic Localization

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- The category-theoretical notion of *symplectic point* seems to be the symplectic seed of Heisenberg indeterminacy principle.

On Symplectic Localization

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- The category-theoretical notion of *symplectic point* seems to be the symplectic seed of Heisenberg indeterminacy principle.

- The (im)possibility of sharply localizing quantum states in phase space depends on the notion of point that we are using:

... while a quantum state cannot be sharply localized at the set-theoretic points of M,...

... it can be sharply localized at its symplectic point...

... given that the symplectic points “internalize” the unsharp variables.

Breaking the Phase Invariance

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- The superposition of two G-phase invariant states $|p_i\rangle$ and $|p_j\rangle$ transforming in different unirreps of G,...

... is no longer G-phase invariant...

... since the G-action changes the relative phases between the two terms $|p_i\rangle + |p_j\rangle$.

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Breaking the Phase Invariance

- The superposition of two G-phase invariant states $|p_i\rangle$ and $|p_j\rangle$ transforming in different unirreps of G,...

  ... is no longer G-phase invariant...

  ... since the G-action changes the relative phases between the two terms $|p_i\rangle + |p_j\rangle$.

- Therefore, ***by introducing an "indeterminacy" in the value of the variable p*** that labels the unirreps of G,...

  ... ***we break the G-phase invariance***,...

  ... **i.e. the complete indeterminacy in the variable q** acted upon by G.

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Gauge Group Actions vs...

A gauge theory is restricted to a single value of μ (namely 0)

$\Downarrow$

The quantum theory only contains G-invariant quantum states

(i.e. states transforming in the trivial unirrep of G)

$\Downarrow$

Since we do not have different unirreps to superpose...

$\Downarrow$

... the gauge invariance cannot be broken.

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... Phase Group Actions

Phase observables are *not* restricted to a single value of μ

$\Downarrow$

The quantum theory contains (G, ξ)-phase invariant states *for all* $\xi \in \mathfrak{g}^*_{\mathbb{Z}}$

(i.e. states transforming in different unirreps of G)

$\Downarrow$

We can superpose (G, ξ)-phase invariant states defined by different unirreps ξ.

$\Downarrow$

The G-phase invariance is broken for such superposed states.

Cohomological Reduction

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The Dirac observables $f \in \mathcal{C}^{\infty}(M_0)$ of a gauge theory can be recovered...

... by means of the BRST-cohomological reformulation of the 0-symplectic reduction.

♣ Kostant, B. & Sternberg, S. [1987]: "Symplectic Reduction, BRS Cohomology, and Infinite Dimensional Clifford Algebras," Annals of Physics 176, 49-113.

Gauge Theories in a Diagram

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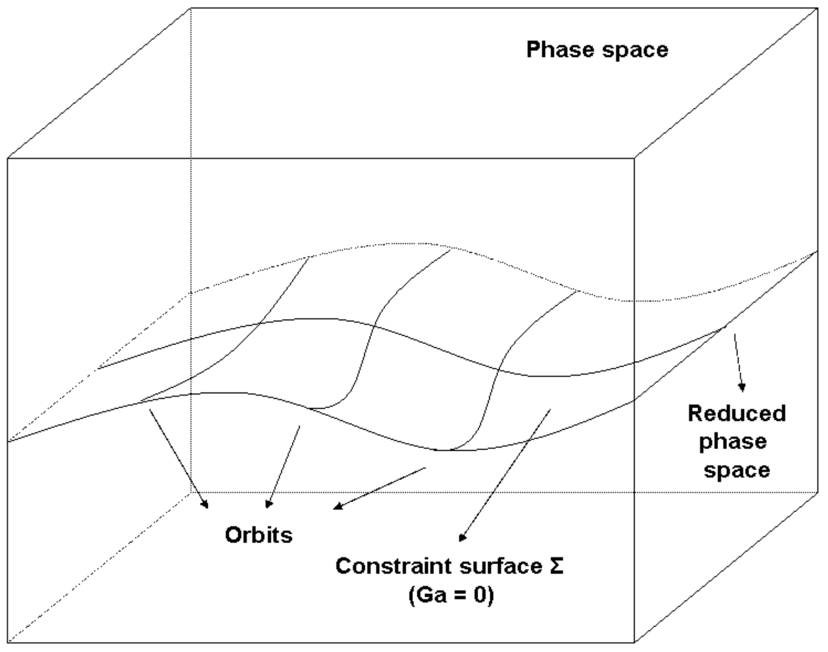

The reduction from M to $M_0 \doteq \Sigma/G$ is a two-step procedure:

.**Restriction** from M to $\Sigma \rightsquigarrow$ **Koszul resolution of** $\mathcal{C}^{\infty}(\Sigma)$.

.**Projection** from Σ to $\Sigma/G \rightsquigarrow$ **Lie algebra cohomology of** $\mathfrak{g}$ **with values in the** $\mathfrak{g}$**-module** $\wedge\mathfrak{g} \otimes \mathcal{C}^{\infty}(M)$.

Koszul Resolution

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- We can describe the algebra of observables on Σ

$$\mathcal{C}^{\infty}(\Sigma) = \mathcal{C}^{\infty}(M)/\langle G_a \rangle$$

in homological terms by extending the (co-moment) map

$$\begin{array}{ccc} \mathfrak{g} & \to & \mathcal{C}^{\infty}(M) \\ X_i & \mapsto & f_i(m) = \langle \mu(m), X_i \rangle \end{array}$$

to the quasi-acyclic complex

$$\ldots \wedge^q \mathfrak{g} \otimes \mathcal{C}^{\infty}(M) \to \wedge^{q-1} \mathfrak{g} \otimes \mathcal{C}^{\infty}(M) \to \ldots \to \mathfrak{g} \otimes \mathcal{C}^{\infty}(M) \xrightarrow{\delta_1} \mathcal{C}^{\infty}(M) \xrightarrow{\delta_0} 0,$$

where the Koszul differential is defined by

$$\delta(X_i \otimes 1) = 1 \otimes f_i, \qquad \delta(1 \otimes f) = 0.$$

Koszul Resolution

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$$\ldots \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(M) \rightarrow \wedge^{q-1}\mathfrak{g} \otimes \mathcal{C}^\infty(M) \rightarrow \ldots \rightarrow \mathfrak{g} \otimes \mathcal{C}^\infty(M) \xrightarrow{\delta_1} \mathcal{C}^\infty(M) \xrightarrow{\delta_0} 0,$$

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$$\delta(X_i \otimes 1) = 1 \otimes f_i, \qquad \delta(1 \otimes f) = 0.$$

- Hence,

$$\begin{aligned} H_0^\delta(\wedge \mathfrak{g} \otimes \mathcal{C}^\infty(M)) &= Ker(\delta_0)/Im(\delta_1) \\ &= \mathcal{C}^\infty(M)/\langle G_a \rangle \\ &= \mathcal{C}^\infty(\Sigma). \end{aligned}$$

Lie Algebra Cohomology

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- Given the $\mathfrak{g}$-module $K = \wedge\mathfrak{g} \otimes \mathcal{C}^{\infty}(M)$, we can define the ***vertical differential*** (or ***Chevalley-Eilenberg differential***)

$$d : K \to \mathfrak{g}^{*} \otimes K = Hom(\mathfrak{g}, K)$$

given by

$$(dk)(X) = X \cdot k, \qquad X \in \mathfrak{g}, k \in K.$$

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- This can be extended to a map

$$d : \wedge^{q}\mathfrak{g}^{*} \otimes K \to \wedge^{q+1}\mathfrak{g}^{*} \otimes K,$$

by

$$d(\eta \otimes k) = d\eta \otimes k + (-1)^{p}\eta \otimes dk, \qquad \eta \in \wedge^{p}\mathfrak{g}^{*}.$$

Lie Algebra Cohomology

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- The 0-Lie algebra cohomology of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module K is given by

$$H^0_d(\wedge\mathfrak{g}^* \otimes K) = K^{\mathfrak{g}}.$$

BRST Cohomology

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- We can then form the double complex

$$\begin{array}{ccc} \wedge^p \mathfrak{g}^* \otimes \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(M) & \xrightarrow{\ \delta\ } & \wedge^p \mathfrak{g}^* \otimes \wedge^{q-1} \mathfrak{g} \otimes \mathcal{C}^\infty(M) \\ \Big\downarrow d & & \\ \wedge^{p+1} \mathfrak{g}^* \otimes \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(M) & & \end{array}$$

with

$$\delta^2 = 0 \qquad d^2 = 0 \qquad \delta d = d\delta$$

such that

$$H_d^0(H_0^\delta(\wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g} \otimes \mathcal{C}^\infty(M))) = \mathcal{C}^\infty(M_0).$$

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BRST Cohomology

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$$\begin{array}{ccc} \wedge^p \mathfrak{g}^* \otimes \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(M) & \xrightarrow{\ \delta\ } & \wedge^p \mathfrak{g}^* \otimes \wedge^{q-1} \mathfrak{g} \otimes \mathcal{C}^\infty(M) \\ \downarrow d & & \\ \wedge^{p+1} \mathfrak{g}^* \otimes \wedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(M) & & \end{array}$$

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$$H_d^0(H_0^\delta(\wedge \mathfrak{g}^* \otimes \wedge \mathfrak{g} \otimes \mathcal{C}^\infty(M))) = \mathcal{C}^\infty(M_0).$$

- Hence, the ***Dirac observables*** in $\mathcal{C}^\infty(M_0)$ can be described as elements f_0^0 that are d-closed modulo δ:

$$df_0^0 = -\delta f_1^1 \approx_\Sigma 0.$$

Pre-Quantum Geometry in a Nutshell

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- Given a Hamiltonian G-manifold (M, ω, μ) such that ω satisfies the ***integrality condition*** (c.f. Kostant, Souriau, Kirillov) according to which $[\omega]$ is in the image of the map

$$H_{\text{Cech}}(M, \mathbb{Z}) \to H_{\text{De Rahm}}(M)$$

... there exists a complex line bundle $L \to M$ with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and a compatible connection ∇ such that

$$curv(\nabla) = 2\pi i \omega.$$

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- The sections $\psi \in \Gamma(L)$ define the so-called ***pre-quantum states***.

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- The sections $\psi \in \Gamma(L)$ define the so-called ***pre-quantum states***.

- The Lie algebra elements $X_i \in \mathfrak{g}$ act on these sections by means of the operators

$$\hat{v}_i = -i\hbar \nabla_{v_{f_i}} + f_i.$$

Polarizations in a Nutshell

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- Since the pre-quantum states can be sharply localized in M, they do not satisfy Heisenberg indeterminacy principle.

Polarizations in a Nutshell

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- Since the pre-quantum states can be sharply localized in M, they do not satisfy Heisenberg indeterminacy principle.

- The pre-quantum geometry has to be enriched by choosing a ***polarization*** $\mathcal{P}$, i.e. an involutive Lagrangian subbundle of TM.

Polarizations in a Nutshell

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- Since the pre-quantum states can be sharply localized in M, they do not satisfy Heisenberg indeterminacy principle.
- The pre-quantum geometry has to be enriched by choosing a ***polarization*** $\mathcal{P}$, i.e. an involutive Lagrangian subbundle of TM.
- We can then “cut in half’ the space of pre-quantum states by means of an eq. of the form

$$\nabla_{\mathcal{P}}\psi = 0.$$

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- Now, we have argued that the reduction in the N$^\circ$ of observables that we need in order to define a state might be understood as a consequence of the G-action.
- Hence, we can conjecture that the group action induces a sort of natural "group-polarization" of the pre-quantum states.

Universal Constraint Surface

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- We do not want to quantize the single 0-symplectic quotient M_0 as in gauge theories...

  ... but rather all the ξ-symplectic quotients M_ξ at once.

Universal Constraint Surface

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- We do not want to quantize the single 0-symplectic quotient M_0 as in gauge theories...

  ... but rather all the ξ-symplectic quotients M_ξ at once.

- Hence, instead of considering a single ξ-constraint surface $\Sigma_\xi = \mu^{-1}(\xi)/G$...

  ... we shall consider the ***universal constraint surface***

$$\Sigma = \Phi^{-1}(0) \subset M \times \mathfrak{g}^*_-,$$

where $M \times \mathfrak{g}^*_-$ is endowed with the product Poisson structure and the shifted moment map

$$\begin{aligned} \Phi : M \times \mathfrak{g}^*_- &\rightarrow \mathfrak{g}^* \\ (m, \xi) &\mapsto \Phi(m,\xi) = \mu(m) - \xi \end{aligned}$$

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$$\begin{aligned} \Phi : M \times \mathfrak{g}^*_- &\rightarrow \mathfrak{g}^* \\ (m, \xi) &\mapsto \Phi(m, \xi) = \mu(m) - \xi \end{aligned}$$

- The surface Σ is defined by the common zeros of the involutive ***universal constraints***

$$G_i(m, \xi) = f_i(m) - \langle \xi, X_i \rangle, \qquad X_i \in \mathfrak{g}$$

$$\{G_i, G_j\} = c^k_{ij} G_k.$$

Shifted Pre-Quantum Geometry

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- Following Guillemin & Sternberg (1982), we can introduce the ***shifted bundle***

$$L_M \boxtimes L^*_{\mathfrak{g}^*} \doteq \pi^*_M L_M \otimes \pi^*_{\mathfrak{g}^*_-} L^*_{\mathfrak{g}^*}$$

defined by the diagram

$$\begin{array}{ccccc} L_M & & L_M \boxtimes L^*_{\mathfrak{g}^*} & & L^*_{\mathfrak{g}^*} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xleftarrow[\pi_M]{} & M \times \mathfrak{g}^*_- & \xrightarrow[\pi_{\mathfrak{g}^*_-}]{} & \mathfrak{g}^*_- \end{array}$$

and endowed with the ***vertical differential ∇ acting along the G-orbits***

$$\nabla_{(m,\xi)} = \nabla^M_m \otimes id + id \otimes \nabla^{\mathcal{O}}_\xi, \qquad \xi \in \mathcal{O} \subset \mathfrak{g}^*$$

which is flat on Σ:

$$F(v_i, v_j)(m, \xi) = c^k_{ij}(f_k(m) - \langle \xi, X_k \rangle) \approx_\Sigma 0.$$

Weak “Group-Polarization” Condition

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- The BRST construction applied to this setting yields in degree 0 the sections of $L_M \boxtimes L^*_{\mathfrak{g}^*}$ whose restriction to Σ is $\mathfrak{g}$-invariant...

  ... i.e. the sections that are ∇-closed modulo δ:

$$\nabla_i \Psi(m,\xi) = \varphi_i^j G_j(m,\xi) \approx_\Sigma 0.$$

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  ... i.e. the sections that are ∇-closed modulo δ:

$$\nabla_i \Psi(m,\xi) = \varphi_i^j G_j(m,\xi) \approx_\Sigma 0.$$

- If we consider the (distribution) sections whose restrictions to Σ are supported by the elements $(\mu^{-1}(\xi),\xi) \in \Sigma$ for a fixed $\xi \in \mathfrak{g}^*$, the cocycle eq. becomes

$$\nabla^M_{v_i} \psi(m) \approx_{(\mu^{-1}(\xi),\xi)} 0.$$

Weak "Group-Polarization" Condition

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- The BRST construction applied to this setting yields in degree 0 the sections of $L_M \boxtimes L^*_{\mathfrak{g}^*}$ whose restriction to Σ is $\mathfrak{g}$-invariant...

  ... i.e. the sections that are ∇-closed modulo δ:

$$\nabla_i \Psi(m,\xi) = \varphi_i^j G_j(m,\xi) \approx_\Sigma 0.$$

- If we consider the (distribution) sections whose restrictions to Σ are supported by the elements $(\mu^{-1}(\xi),\xi) \in \Sigma$ for a fixed $\xi \in \mathfrak{g}^*$, the cocycle eq. becomes

$$\nabla^M_{v_i} \psi(m) \approx_{(\mu^{-1}(\xi),\xi)} 0.$$

- By using the pre-quantum operators

$$\hat{v}_f = -i\hbar \nabla_{v_f} + f$$

  this eq. can be rewritten as an eigenvalue eq.

$$\begin{aligned} \hat{v}_i \psi(m) &\approx f_i(m)\psi(m), \\ &\approx \langle \mu(m), X_i \rangle \psi, \\ &= \langle \xi, X_i \rangle \psi. \end{aligned}$$

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Conclusions

- We have argued that ***phase symmetries*** and ***gauge symmetries***...

... are different manifestations of the same geom. formalism, i.e. the ***Mardsen-Weinstein symplectic reduction***.

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- ***From a technical viewpoint***, this facts points towards the possibility of a ***BRST cohomological quantization*** of ordinary (non-constrained) theories...

  ... in which the "polarization" of quantum states naturally arises from the condition of $\mathfrak{g}$-invariance on the cocycles.

This is the End

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Many thanks for your kind attention !!!