Renormalization in Tensorial Group Field Theory

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  - **Group Field Theories**: generating functionals for Spin Foam Models, and 2nd quantized version of canonical Loop Quantum Gravity.
  - **Tensorial**: prescription for the combinatorial structure of the interactions, coming from recent work in Tensor Models. [Gurau, Rivasseau, Bonzom, Ryan, ...]

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- Several perturbatively renormalizable TGFTs on the market:
  - 4d combinatorial model on $\mathrm{U}(1)$, with Laplace operators in the covariance [Ben Geloun, Rivasseau '11];
  - 3d combinatorial model on $\mathrm{U}(1)$, with first-order derivative operators in the covariance [Ben Geloun, Rivasseau '11];
  - $\mathrm{U}(1)$ models with gauge invariance condition, and Laplace operators in the covariance [Oriti, Rivasseau, SC '12] [Ousmane Samary, Vignes-Tourneret '12] [Ousmane Samary '13];
  - 3d $\mathrm{SU}(2)$ model with gauge invariance condition, and Laplace operators in the covariance [Oriti, Rivasseau, SC '13];
  - and more, with various kinds of derivative operators in the propagators... [Ben Geloun, Livine '12] [Ben Geloun '13]

Motivations

- Make GFTs well-defined as (perturbative) QFTs.
- Develop tools to analyze the phase diagrams of such theories. What is the effective dynamics of the many-body sector of LQG/SF?
- In particular, develop a Wilsonian RG picture: classification of relevant and irrelevant interactions near the Gaussian fixed point, investigate the existence of other fixed points, etc.

Outline

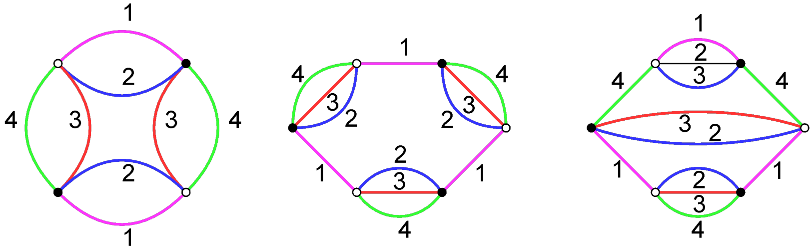
1. Rank-3 TGFT model on $\mathrm{SU}(2)$
2. Renormalization group flow equations
3. Properties of the Gaussian fixed point
4. TGFT in $D = 4 - \varepsilon$: first hints of a non-trivial fixed point?

Definition: colored graph

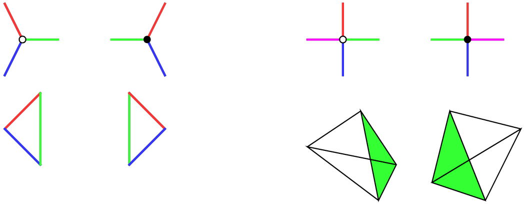
A ***n*-colored graph** is a **bipartite regular** graph of valency n, **edge-colored** by labels $\ell \in \{1, \ldots, n\}$, and such that at each vertex meet n edges with distinct colors.

- Two types of nodes: black or white dots.
- n types of edges, with color label $\ell \in \{1, \ldots, n\}$.

Examples: 4-colored graphs.



- Each node in a $(d+1)$-colored graph is dual to a *d*-**simplex**



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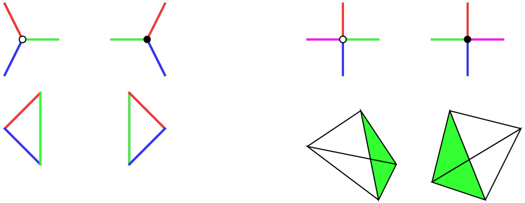
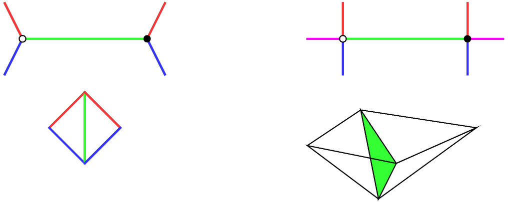


- Each line represents the **gluing** of two d-simplices along their boundary **$(d-1)$-simplices**



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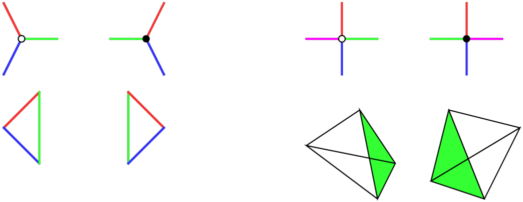

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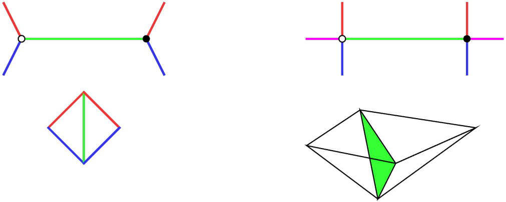


$\Rightarrow$ **A $(d+1)$-colored graph represents a triangulation in dimension d.**

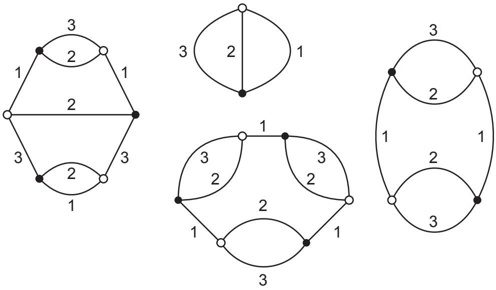
3d discrete gravity model

- Euclidean gravity in 3d $=$ BF theory
- To make sense of the formal quantum gravity path integral, one discretizes it on a cellular complex:

$$\mathcal{Z}_{\mathcal{M}} = \int [\mathcal{D}\omega]\delta(F(\omega)) \to \mathcal{Z}_{\triangle} = \int [\mathrm{d}h_l] \prod_{f \in \triangle^*} \delta\left(\overrightarrow{\prod_{l \in f}} h_l\right)$$

- Traditional construction: $\triangle$ $=$ simplicial complex $\Rightarrow$ **Ponzano-Regge spin foam model**.
- Instead, one can use a **colored cellular complex**.
- Main advantages:
  - full **homology**;
  - good control over the **topology**;
  - tools from $1/N$ **expansion** [Gurau '10, ...].

- Elementary building blocks = 3-**bubbles** = **3d cells with colored triangulated boundaries**...



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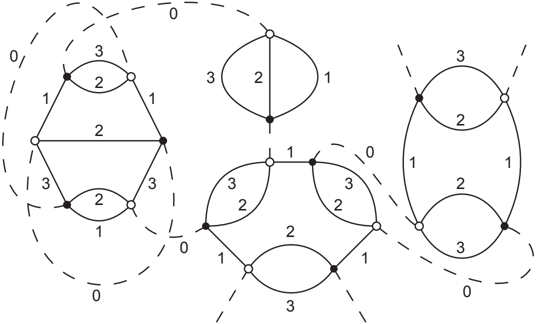


- ...glued together along their boundary triangles.
- **Holonomy variables** associated to the dashed, color-0 lines.
- **Face of color** ℓ = connected set of (alternating) color-0 and color-ℓ lines.

Summing over graphs

- Two possible strategies to define the continuum limit: **refining** or **summing**.
- GFT strategy: summing using the QFT formalism.
- **Dynamical variable**: rank-3 complex field

$$\varphi : \mathrm{SU}(2)^3 \ni (g_1, g_2, g_3) \mapsto \mathbb{C}\,.$$

- **Partition function**:

$$\mathcal{Z} = \int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \mathrm{e}^{-S(\varphi,\overline{\varphi})}\,.$$

  - $S(\varphi, \overline{\varphi})$ is the interaction part of the action, and should be a sum of **connected tensor invariants**

$$S(\varphi, \overline{\varphi}) = \sum_{b \in \mathcal{B}} t_b I_b(\varphi, \overline{\varphi})$$

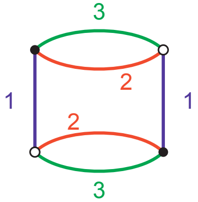
  which play the role of **local** terms.

  - **Closure constraint** or **gauge invariance condition** imposed by the Gaussian measure $\mathrm{d}\mu_C$:

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int \mathrm{d}h \prod_{\ell=1}^{3} \delta\left(g_\ell h g_\ell'^{-1}\right)\,.$$

Correspondence between colored graphs b and tensor invariants $I_b(\varphi, \overline{\varphi})$:

- white (resp. black) **node** $\leftrightarrow$ **field** (resp. complex conjugate field);
- **edge** of color $\ell \leftrightarrow$ **convolution** of ℓ-th indices of φ and $\overline{\varphi}$.



$$\int [\mathrm{d}g_i]^6 \, \varphi(g_1, g_2, g_3)\overline{\varphi}(g_1, g_4, g_5)\varphi(g_6, g_4, g_5)\overline{\varphi}(g_6, g_2, g_3)$$

Divergences and scales

- The amplitudes are **generically divergent**.
- Two strategies to cure the divergences: $1/N$ **expansions** or **renormalization**.
- **Problem**: no notion of scale, the covariance is a projector (analogue of an ultralocal QFT).
- **Solution**: compose the original projector with a **non-trivial differential operator**. For instance

$$\left(m^2 - \sum_{\ell=1}^{d} \Delta_\ell \right)^{-1} ,$$

  which is a conservative choice also suggested by study of radiative corrections [Ben Geloun, Bonzom '11].

This defines the new Gaussian measure $\mathrm{d}\mu_C$:

$$\int \mathrm{d}\mu_C(\varphi, \overline{\varphi})\, \varphi(g_\ell)\overline{\varphi}(g'_\ell) = C(g_\ell; g'_\ell) = \int_0^{+\infty} \mathrm{d}\alpha\, \mathrm{e}^{-\alpha m^2} \int \mathrm{d}h \prod_{\ell=1}^{3} K_\alpha(g_\ell h g'^{-1}_\ell) \,,$$

where K_α is the **heat kernel** on $\mathrm{SU}(2)$ at time α.

Slicing of scales: fix $M > 1$ and write

$$C = \int_0^{+\infty} \mathrm{d}\alpha \, \ldots = \sum_{i=0}^{+\infty} C_i \,, \qquad C_i = \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \ldots$$

$\Rightarrow$ **momentum slices** labeled by the integer $i \in \mathbb{N}$.

- Effective action:

$$S_i(\varphi, \overline{\varphi}) = \sum_b t_{b,i} \frac{I_b(\varphi, \overline{\varphi})}{k(b)}$$

- Discrete flow upon integration of the shell $\alpha \in \left[M^{-2i}, M^{-2(i-1)} \right]$:

$$\log \left(\int \mathrm{d}\mu_{C_i}(\varphi, \overline{\varphi}) \, \mathrm{e}^{-S_i(\Phi+\varphi, \overline{\Phi}+\overline{\varphi})} \right) \underset{+\, wave-function\ ren.}{\longrightarrow} S_{i-1}(\Phi, \overline{\Phi})$$

$$t_{b,i} \longrightarrow t_{b,i-1}$$

Power-counting and dimensionless couplings

- **Power-counting** in a slice:

$$|\mathcal{A}_{\mathcal{G}}| \leq K M^{\omega(\mathcal{G}) i} ,$$

where the degree of divergence ω is

$$\omega = 3 - \frac{N}{2} + \sum_{k \in \mathbb{N}} (3 - k) n_{2k} + 3\rho .$$

N = number of external legs.
n_{2k} = number of vertices with valency $2k$.
$\rho = 0$ for a melonic graph, and $\rho \leq -1$ otherwise.

- In order to kill the n_{2k} dependence, we need to introduce **dimensionless coupling constants**

$$u_{b,i} \equiv t_{b,i} M^{-d_b i} , \qquad d_b = [t_b] \equiv 3 - \frac{N_b}{2} ,$$

where N_b is the valency of the bubble b.

Melonic flow equations

- Intermediate coupling constants defined by a sum over melonic graphs:

$$\tilde{u}_{b,i-1} M^{-d_b} = u_{b,i} - \sum_{\mathcal{G} \in \mathcal{M}(b)} \frac{k(b)}{k(\mathcal{G})} \left(\prod_{b'} (-u_{b',i})^{n_{b'}(\mathcal{G})} \right) a(u_{2,i}, \mathcal{G}) \,.$$

- Rescaling due to **wave-function renormalization**:

$$u_{b,i-1} \equiv \frac{\tilde{u}_{b,i}}{(1 + C T_{\varphi,i-1})^{N_b/2}} \,.$$

In this framework, **all melonic graphs** contribute to the flow equations.

Properties of the Gaussian fixed point

Linearization

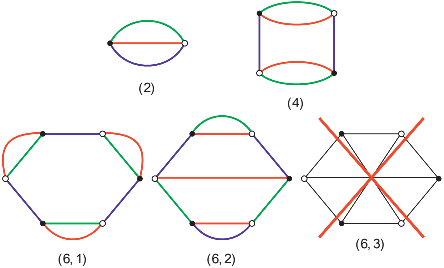
- **Linearized flow equations**:

$$\forall b \in \mathcal{B}\,, \qquad u_{b,i-1} = M^{d_b} \left[u_{b,i} + \sum_{b' > b} \lambda(b, b') u_{b',i} \right] + \mathcal{O}(u^2)\,.$$

- Triangular system of equations, with diagonal $= (M^{d_b})_{b \in \mathcal{B}} \Rightarrow$ (formally) diagonalizable.

- Coupling constants with $N_b > 6$ only contribute to eigendirections with eigenavalues $M^{d_b} < 1$. They are therefore stable directions, or in other words irrelevant.

- Bubbles with $N_b \leq 6$:



$(6,3)$ bubbles completely decouple and can also be ignored.
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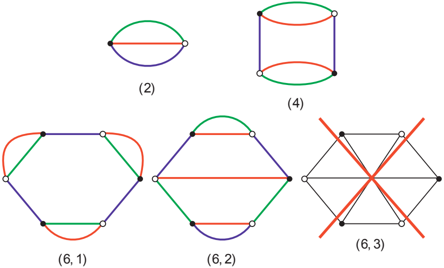

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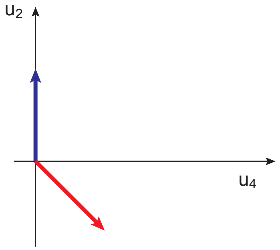
- The linearized flow takes the form:

$$\begin{pmatrix} u_{2,i-1} \\ u_{4,i-1} \\ u_{6,1,i-1} \\ u_{6,2,i-1} \end{pmatrix} = \begin{pmatrix} M^2 & M^2\lambda(2,4) & M^2\lambda(2,(6,1)) & M^2\lambda(2,(6,2)) \\ 0 & M & M\lambda(4,(6,1)) & M\lambda(4,(6,2)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{2,i} \\ u_{4,i} \\ u_{6,1,i} \\ u_{6,2,i} \end{pmatrix}$$

Eigendirections

The **eigendirections** can be computed:

$$\sigma_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \; \sigma_4 = \begin{pmatrix} -6\sqrt{\pi} \\ 1 \\ 0 \\ 0 \end{pmatrix} , \; \sigma_{6,1} = \begin{pmatrix} 6\pi \\ -2\sqrt{2\pi} \\ 1 \\ 0 \end{pmatrix} , \; \sigma_{6,2} = \begin{pmatrix} 15\pi \\ -4\sqrt{2\pi} \\ 0 \\ 1 \end{pmatrix} .$$

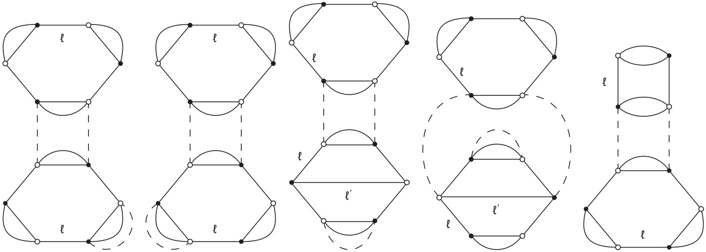


The exponential suppression of the relevant directions in the UV imposes:

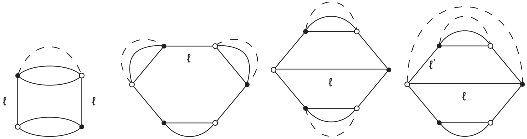
$$u_{2,i} \underset{i \to +\infty}{\approx} 6\pi \, u_{6,1,i} + 15\pi \, u_{6,2,i} \, ,$$

$$u_{4,i} \underset{i \to +\infty}{\approx} -2\sqrt{2\pi} \, u_{6,1,i} - 4\sqrt{2\pi} \, u_{6,2,i} \, .$$

In order to determine the fate of φ^6 **couplings**, one needs to compute the second order.



Contrary to ordinary scalar field theories, the wave-function renormalization plays a crucial role at this order, with contributions from:



Marginal couplings

- One finds:

$$u_{6,1,i-1} = u_{6,1,i} + \gamma_{11}\, {u_{6,1,i}}^2 + \gamma_{12}\, u_{6,1,i} u_{6,2,i} + \gamma_{14}\, u_{6,1,i} u_{4,i} + \mathcal{O}(u^3)\,,$$

  and

$$u_{6,2,i-1} = u_{6,2,i} + \gamma_{21}\, u_{6,1,i} u_{6,2,i} + \gamma_{22}\, {u_{6,2,i}}^2 + \gamma_{24}\, u_{6,2,i} u_{4,i} + \mathcal{O}(u^3)\,.$$

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- Due to wave-function counter-terms at first order: $\gamma_{ij} > 0$.
- But because of the damping of the relevant directions, one has in the deep UV:

$$u_{6,1,i-1} \underset{i \to +\infty}{\approx} u_{6,1,i} \left(1 - \beta_{11}\, u_{6,1,i} - \beta_{12}\, u_{6,2,i}\right) ,$$
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with: $\beta_{ij} > 0$.

Remark. With a continuously varying cut-off Λ, one would have

$$\Lambda \frac{\partial u_{6,1,\Lambda}}{\partial \Lambda} \underset{\Lambda\to+\infty}{\approx} \beta_{11}\, u_{6,1,\Lambda}{}^2 + \beta_{12}\, u_{6,1,\Lambda} u_{6,2,\Lambda}\,,$$
$$\Lambda \frac{\partial u_{6,2,\Lambda}}{\partial \Lambda} \underset{\Lambda\to+\infty}{\approx} \beta_{21}\, u_{6,1,\Lambda} u_{6,2,\Lambda} + \beta_{22}\, u_{6,2,\Lambda}{}^2$$

Marginal couplings

Conclusions:

- Trajectories with $u_{6,1} > 0$ and $u_{6,2} > 0$ are not asymptotically free.
- Despite the fact that wave-function renormalization dominates over coupling-constant renormalization, the final β-coefficients have the same signs as in ordinary scalar field theory: consequence of the presence of a superrenormalizable coupling constant (u_4).

New questions:

- Can we construct asymptotically free trajectories with opposite signs for $u_{6,1}$ and $u_{6,2}$?
- Is there a non-trivial fixed point preventing the existence of a Landau pole, even when $u_{6,1} > 0$ and $u_{6,2} > 0$?
- Are TGFTs with quartic marginal interactions asymptotically free in general?

It is possible to consider similar gauge invariant models with different groups and tensor ranks. Necessary conditions for their renormalizability were derived, in terms of the rank d, the dimension of the group D, and the maximal valency of the relevant bubbles v_{max}:

Type	d	D	v_{max}	ω
A	3	3	6	$3 - N/2 - 2n_2 - n_4 + 3\rho$
B	3	4	4	$4 - N - 2n_2 + 4\rho$
C	4	2	4	$4 - N - 2n_2 + 2\rho$
D	5	1	6	$3 - N/2 - 2n_2 - n_4 + \rho$
E	6	1	4	$4 - N - 2n_2 + \rho$

[Oriti, Rivasseau, SC '13]

- $d = D = 3$ is the only case for which the combinatorial dimension can match the dimension of space-time inferred from the symmetry group G.

- Analogy with ordinary scalar field theory: at fixed $d = 3$
  - φ^6 model in $D = 3$;
  - φ^4 model in $D = 4$.

- $D = 4$: **asymptotic freedom** of trajectories with $u_4 > 0$ (quartic model)
- One way of analytically continue the group dimension D

$$\mathrm{SU}(2) \mapsto \mathrm{SU}(2) \times \mathrm{U}(1)^{1-\varepsilon}$$

- New **non-trivial fixed point**:

$$u_2^* \sim a\,\varepsilon + \mathcal{O}(\varepsilon^2)\,, \qquad u_4^* \sim -b\,\varepsilon + \mathcal{O}(\varepsilon^2)\,,$$

  with $a, b > 0$.

- Opposite signs with respect to the Wilson-Fischer fixed point in ordinary scalar field theories.
- Does it survive in $D = 3$? Indication of asymptotic safety of the trajectories with $u_{6,1} > 0$ and $u_{6,2} > 0$?

Summary

- **Perturbatively renormalizable** TGFT models exist, despite the non-local nature of these theories.
- Progress towards theories which are more and more relevant to LQG and spin foams: $\mathrm{SU}(2)$ group, closure constraint.
- **Asymptotic freedom** can be realized in such models, especially when only quartic interactions are renormalizable.
- First hints of **non-trivial fixed point** and asymptotic safety in a φ^6 model.

More generally: Renormalization in GFT is imposing itself as a central tool, which hopefully will help addressing key difficulties, especially in four dimensions: **continuum limit**, **phase transitions**, **universality** etc.

$\Rightarrow$ efforts in this direction should be pursued: FRGE, non-compact groups, simplicity constraints...

Thank you for your attention