Noncommutative version of Borcherds’ approach to QFT

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Why is NCG not highly developed by physicists?
- Compact manifold instead of globally hyperbolic
- Riemannian instead of Lorentzian
- This is not a quantum field theory

Quantization starts from a commutative algebra (normal products, functionals)

The tensor product of algebras is natural

How to quantize the tensor product and recover standard QFT?

We need a cocommutative Hopf algebra
OUTLINE

- Construction of a quantum field theory
  - Classical field theory (Lagrangian)
  - Many-body algebra (product of Lagrangians)
  - Quantization
  - Renormalization

- Algebraic functional approach

- Borcherds’ approach: Hopf algebra bundle

- How to extend this to the noncommutative case?
FUNCTIONAL APPROACH

- Algebraic functional framework (R. Brunetti, M. Dütsch, K. Fredenhagen, K. Rejzner)
- Configuration space: $\mathcal{E} = \Gamma(M, E)$ space of smooth sections of a vector bundle $E \xrightarrow{\pi} M$
- Functionals $C^\infty(\mathcal{E}, \mathbb{R})$, action, observables, etc.
- Deformation quantization (wavefront set argument)
- Renormalization by causality (Stueckelberg, Bogoliubov, Epstein, Glaser, Stora, etc.)
The aim of this paper is to describe how to use regularization and renormalization to construct a perturbative quantum field theory from a Lagrangian. We first define renormalizations and Feynman measures, and show that although there need not exist a canonical Feynman measure, there is a canonical orbit of Feynman measures under renormalization. We then construct a perturbative quantum field theory from a Lagrangian and a Feynman measure, and show that it satisfies perturbative analogues of the Wightman axioms, extended to allow time-ordered composite operators over curved spacetimes.
Not a bad mathematician (Fields medal 1998)
Has been working 10 years on QFT
Original point of view
Not easy to read (no context)
Contains some mistakes
Gives the same results as standard QFT
Hopf algebraic and geometric interpretation of QFT
Fruitful generalizations
- Classical field theory (Lagrangians)
- Many-body algebra
- Quantization
- Renormalization
Borcherds’ Approach

- Vector bundle $E \xrightarrow{\pi} M$

- Configuration space $\Gamma(M, E)$ (classical fields)

- Derivative of fields are sections $\Gamma(M, JE)$ of the jet bundle $J^\infty E \xrightarrow{\pi} M$

- Lagrangians are sections $\Gamma(M, S(JE^*))$ of the Hopf algebra bundle $S(J^\infty E^*) \xrightarrow{\pi} M$
THE LAGRANGIAN

- Functional approach: the action is a local functional
  \[ I(\varphi) = \int_M f(j_x \varphi) g(x) d\mu_x \]
  \[ f(j_x \varphi) = \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{1}{2} m^2 \varphi^2(x) + \frac{\lambda}{4!} \varphi^4(x) \]

- Borcherds’ approach: the Lagrangian density is a polynomial in the field and its derivatives
  \[ L \in \Gamma(M, S(JE^*)) \cong \mathcal{S}_{C^\infty(M)}(\Gamma(M, JE^*)) \]

- Correspondence: \( L(\varphi)(x) = f(j_x \varphi) g(x) \)
  \[ I(\varphi) = \int_M L(\varphi)(x) d\mu_x \]
Almost commutative geometry

- Configuration space $\mathcal{C}^\infty(M, \mathcal{A}_F) \cong \Gamma(M, E)$
  where $E \cong M \times \mathcal{A}_F$ is an algebra bundle (Boeijink and Suijlekom)

- $\mathcal{C}^\infty(M, \mathcal{A}_F)$ is an algebra over $\mathcal{C}^\infty(M)$
- Any algebra is an algebra over its center
- **Standard model**: The finite algebra $\mathcal{A}_F$ is a cocommutative Hopf algebra
OUR APPROACH

- The noncommutative Lagrangian will belong to the space $\Gamma(M, E_H)$ of sections of the cocommutative Hopf algebra bundle $E_H \xrightarrow{\pi} M$

- Locally $E_H|_U = U \times H$

- For Borcherds $E_H$ is the commutative and cocommutative Hopf algebra $S(JE^*)$

- The space $\Gamma(M, E_H)$ of Lagrangians is a cocommutative Hopf algebra over $C^\infty(M)$
QFT BUILDING

- Classical field theory
- Many-body algebra (product of Lagrangians)
- Quantization
- Renormalization
THE S-MATRIX

- The scattering matrix
  \[ S = T(e^{\frac{i}{\hbar} I(\varphi)}) \]
  \[ S = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n} \int_{M^n} T(L(x_1) \ldots L(x_n)) d\mu x_1 \ldots d\mu x_n \]

- The Lagrangian is \( L \in \Gamma(M, E_H) \)

- Borcherds’ many body algebra:
  - \( \mathcal{H} = S_\mathbb{R}(V) \) with \( V = \Gamma(M, S(JE^*)) \)
  - Example \( \varphi^2(x) \partial_\mu \varphi(y) \)
  - Very close to the physical normal product space
FOCK ALGEBRA

- **Problems:**
  - $H = S_{\mathbb{R}}(V)$ with $V = \Gamma(M, S(JE^*))$ is too large
  - $H = S_{\mathbb{R}}(V)$ has no geometric meaning: it is not the space of sections of a vector bundle because in dimension $d > 2$ a symmetrized $M^n$ is not a topological manifold

- **Our approach:**
  - Work with $H_n = \Gamma(M^n, E^*_H)$ (external tensor product)
  - This a Hopf algebra over $C^\infty(M^n)$
  - This is quantum field theory where the normal product is the tensor product: can we recover standard QFT?
  - Quantization is the deformation of a noncommutative product
FOCK ALGEBRA

Problem

- For each $n$, $H_n = \Gamma(M^n, E^n_H)$ is a Hopf algebra over a different ring $R_n = C^\infty(M^n)$

Solution: inductive limit

- For $n \geq m$ we define the map $\phi_{nm} : M^n \to M^m$
  by $\phi_{nm}(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$
- Let $f_m \in R_m$ and $f_n \in R_n$, we say that $f_m \equiv f_n$ if and only if $f_n = f_m \circ \phi_{nm}$
- Then $R = \lim_{\rightarrow} R_n$ is a unital ring
- Every $f \in R$ has a representative $f_m$ in some $R_m$ and then the representative $f_m \circ \phi_{nm}$ in $R_n$ for all $n \geq m$
- This allows for the definition of the product in $R$
FOCK ALGEBRA

\[\phi_{5,3} : M^5 \rightarrow M^3 \quad \phi_{5,3}(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3)\]

\[C^\infty(M^5) \xleftarrow{\phi_{53}^*} C^\infty(M^3)\]

\[(\phi_{53}^* f)(x_1, x_2, x_3, x_4, x_5) = f(x_1, x_2, x_3)\]
FOCK ALGEBRA

- The inductive limit $H = \lim \Gamma(M^n, E_H^{\otimes n})$ is a Hopf algebra over $R$ (Bourbaki *Algebra I*)
- The projective limit $M = \lim M^n$ is an infinite dimensional manifold modelled on $\lim (\mathbb{R}^d)^n$ (Fréchet, nuclear locally convex space)
- The geometric interpretation is preserved because $H$ is the space of sections of a Hopf bundle on the infinite dimensional manifold $M$
- This is a many-body algebra: the Hopfock algebra
The inductive limit uses the elitist maps
\[ \phi_{nm}(x_1, \ldots, x_n) = (x_1, \ldots, x_m) \]
This choice is arbitrary. Why using the first variables instead of the last or any subset?
We could use a more democratic inductive limit, indexed by subsets of \( \mathbb{N} \)
For \( J \supset I \), where \( J = \{j_1, \ldots, j_n\} \) and \( I = \{i_1, \ldots, i_m\} \) we define \( \varphi_{JI}(x_{j_1}, \ldots, x_{j_n}) = (x_{i_1}, \ldots, x_{i_m}) \)
This uncountable inductive limit is the same as before
QFT BUILDING

- Classical field theory
- Many-body algebra
- Quantization (algebra deformation)
- Renormalization
Quantization

- Wick theorem for $\varphi^2(x) \star \varphi^2(y)$ (Feynman diagrams)

\[
\begin{array}{ccc}
  & x & \star & y \\
  \downarrow & & \downarrow & & \downarrow \\
 & x & & y & \\
\end{array}
\]

- Explicit expression

\[
\varphi^2(x) \star \varphi^2(y) = :\varphi^2(x)\varphi^2(y): + 4D_+(x, y) :\varphi(x)\varphi(y): + 2D_+^2(x, y)
\]

- The Wightman propagator

\[
D_+(x, y) = \langle 0|\varphi(x) \star \varphi(y)|0\rangle
\]

- Functional approach (star product)

\[
(F \star G)(\varphi) = e^{\int dxdy D_+(x,y) \frac{\delta^2}{\delta \varphi(x) \delta \psi(y)}} F(\varphi) G(\psi) |_{\psi = \varphi}
\]
Explicit expression

\[ \varphi^2(x) \star \varphi^2(y) = :\varphi^2(x)\varphi^2(y): + 4D_+(x, y) :\varphi(x)\varphi(y): + 2D_+^2(x, y) \]

Quantum group approach (Drinfeld, 1986)

\[ A \star B = \sum (A_{(1)}|B_{(1)}) A_{(2)} B_{(2)} \]

where \( \Delta A = \sum A_{(1)} \otimes A_{(2)} \)

Laplace pairing: \( (\varphi(x)|\varphi(y)) = D_+(x, y) \)

\[ (AB|C) = \sum (A|C_{(1)})(B|C_{(2)}) \]

The Hopf algebra is here the Hopfock algebra
Quantization is carried out by using the Wightman propagator \( D_+(x, y) \), which is not a smooth function.

More precisely \( D_+ \in \mathcal{D}'(M^2, E^{\bigotimes 2}) \).

Structure theorems for \( \mathcal{D}'(M^2, E^{\bigotimes 2}) = \left( \Gamma_c(M^2, (E^*)^{\bigotimes 2}) \right)' \)

\[
\mathcal{D}'(M^2, E^{\bigotimes 2}) \cong \mathcal{D}'(M^2) \otimes_{C^\infty(M^2)} \Gamma(M^2, E^{\bigotimes 2})
\]

\[
\mathcal{D}'(M^2, E^{\bigotimes 2}) \cong \mathcal{L}_{C^\infty(M^2)}(\Gamma(M^2, (E^*)^{\bigotimes 2}), \mathcal{D}'(M^2))
\]

We quantize the space of distributional sections

\[
\mathcal{M}_n = \mathcal{D}'(M^n, E^{\bigotimes n}_H) = \left( \Gamma_c(M^n, (E^*_H)^{\bigotimes n}) \right)'
\]
As a space of distributions, $\mathcal{M}_n$ cannot be a Hopf algebra

But $\mathcal{M}_n$ is a Hopf module over $\underline{H}_n = \Gamma(M^n, E_{\underline{H}}^\otimes)$

\[ \mathcal{M}_n \cong \mathcal{D}'(M^n) \otimes_{\mathcal{C}^\infty(M^n)} \underline{H}_n \]

The space of coinvariants of $\mathcal{M}_n$ is $\mathcal{D}'(M^n)$, which is a partial algebra (Hörmander’s condition)

The Laplace pairing becomes a map

\[ (\cdot|\cdot) : \underline{H}_n \otimes_{R_n} \underline{H}_n \to \mathcal{D}'_\gamma(M^n) \quad R_n = \mathcal{C}^\infty(M^n) \]

\[ (AB|C) = \sum (A|C_{(1)})(B|C_{(2)}) \]
THE WAVEFRONT SET

Wightman propagator

Feynman propagator
We are now ready to define a star product on a modified $\mathcal{M}_n \cong \mathcal{D}'(M^n) \otimes_{\mathcal{R}_n} H^n$, where $\lambda$ is now an open cone.

We denote the coaction $\beta : \mathcal{M}_n \rightarrow \mathcal{M}_n \otimes_{\mathcal{R}_n} H^n$ by the Sweedler’s notation $\beta(A) = \sum A' \otimes_{\mathcal{R}_n} A''$.

The star product on $\mathcal{M}_n$ is

$$A \star B = \sum A'B'(A''|B'')$$

This product is associative.

Extension to $\mathcal{M} = \lim_{\rightarrow} \mathcal{M}_n$
- Classical field theory
- Many-body algebra
- Quantization
- Renormalization (time-ordered product)
According to the Stueckelberg-Bogoliubov-Epstein-Glaser-Stora-Brunetti-Fredenhagen approach, the time ordered-product is almost entirely a consequence of causality.

For any \((x_1, \ldots, x_n) \in M^n\) which does not belong to the small diagonal \(D_n = \{(x_1, \ldots, x_n); x_1 = \cdots = x_n\}\) there is \(I \subset \{1, \ldots, n\}\) such that \(x_i\) is not in the past causal cone of \(x_j\) if \(i \in I\) and \(j \notin I\).

Causality equation: over \(C_I\)

\[
T(\sigma_1 \otimes \cdots \otimes \sigma_n) = \phi_I^* \left( T\left( \bigotimes_{i \in I} \sigma_i \right) \right) \ast \phi_{I^c}^* \left( T\left( \bigotimes_{j \in I^c} \sigma_j \right) \right)
\]
\[
\begin{align*}
C_{\{3,5\}} & \quad (x_1, x_2, x_3, x_4, x_5) \in M^5 \\
\phi_{\{3,5\}} & \quad x_3 \not\subseteq x_1, \quad x_3 \not\subseteq x_2, \quad x_3 \not\subseteq x_4 \\
\phi_{\{1,2,4\}} & \quad x_5 \not\subseteq x_1, \quad x_5 \not\subseteq x_2, \quad x_5 \not\subseteq x_4
\end{align*}
\]

\[T(\sigma_1 \otimes \cdots \otimes \sigma_5) = \left(\phi_{\{3,5\}}^* \circ T(\sigma_3 \otimes \sigma_5)\right) \ast \left(\phi_{\{1,2,5\}}^* \circ T(\sigma_1 \otimes \sigma_2 \otimes \sigma_4)\right)\]
The time-ordered product $T : H^n \rightarrow M_n$ is a comodule morphism

$$\beta \circ T = (T \otimes \text{Id})\Delta$$

This is equivalent to the Wick expansion

$$T(A) = \sum t(A_{(1)})A_{(2)} \quad t(A) = \varepsilon(T(A))$$

$$T(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_n}) = \sum_{i_1,\ldots,i_n} \binom{k_1}{i_1} \ldots \binom{k_n}{i_n} \langle 0 | T(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_n}) | 0 \rangle \varphi_{k_1-i_1} \otimes \ldots \otimes \varphi_{k_n-i_n}$$

This recursively defines the time-ordered product up to the thin diagonal.

The distributions $t(A)$ are then extended to the thin diagonal (overall renormalization)
Renormalization ambiguity

\[ T'(A) = \sum e^{*\lambda}(A_{(1)})T(A_{(2)}) \]

There is a large renormalization group

The RG is the same as in the functional approach, except that we do not require symmetry

For the special case of \( E_H = S(JE^*) \), we recover exactly the results of standard renormalized perturbative QFT
The space $C^\infty (M, \mathcal{A}_F) \cong \Gamma (M, E)$ does not contain the Lagrangian

Spectral action

\[
I(A) = \text{Tr} \ g \left( \frac{D_A^2}{\Lambda^2} \right)
\]

Naive quantization is not possible (as usually)
CONCLUSION

- Second quantization and renormalization of a cocommutative Hopf algebra bundle
- Standard QFT is recovered
- Noncommutative version of BRST and BV
- BRST for noncommutative geometry (see Roberta Iseppi and Walter van Suijlekom)
- Causality beyond space-time (Fabien Besnard)
- Hopf algebra
\[ \Delta \varphi^m(x) \varphi^n(y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \varphi^i(x) \varphi^j(y) \otimes \varphi^{m-i}(x) \varphi^{n-j}(y) \]
\[ \varepsilon\left((\partial^\alpha \varphi(x))^n\right) = \delta_{n,0} \delta(x) \]

- Laplace pairing (coquasitriangular structure) well defined
\[ (\varphi^m(x) | \varphi^n(y)) = \delta_{n,m} n! D^n_+(x, y) \]

- Wick’s theorem
\[ \varphi^m(x) \star \varphi^n(y) = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} k! D^k_+(x, y) \varphi^{m-k}(x) \varphi^{n-k}(y) \]