Deformation Quantization

FFP14
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References

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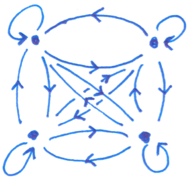

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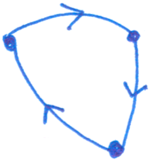

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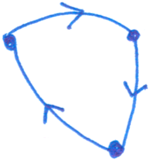

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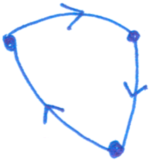

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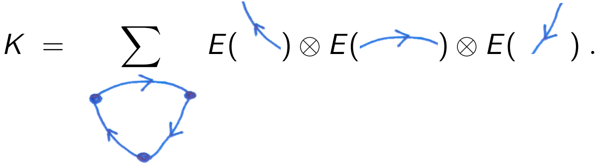
A natural basis of $\mathbb{A}$ is given by the characteristic functions of arrows:

$$E(\rightarrow)(x) \;:=\; \begin{cases} 1 & \text{if } x = \rightarrow \\ 0 & \text{otherwise} \end{cases}$$

Matrices and triangles

Then:



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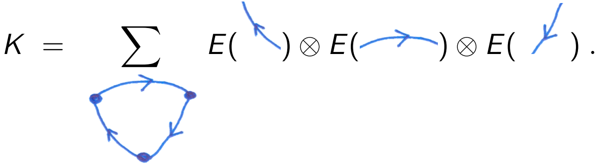

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$\Rightarrow$

Would $E(\quad) \otimes E(\quad) \otimes E(\quad)$ correspond to an exponential??

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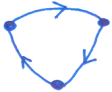
remind:  $\rightsquigarrow E($  $) \otimes E($  $) \otimes E($  $)$

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In other words, to observables $a, b \in C_c^\infty(\mathbb{R}^{2n})$, one associates:

$$a \star_\hbar b(x) \ := \ \frac{1}{\hbar^{2n}} \int\int K_\hbar(x,y,z)\, a(y)\, b(z)\, \mathrm{d}y\, \mathrm{d}z$$

where

$$K_\hbar(x,y,z) \ = \ e^{\frac{i}{\hbar}(\omega(x,y)+\omega(y,z)+\omega(z,x))} \qquad (\omega(x,y) := \omega_{ij} x^i y^j)$$

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Asymptotics:

$$a \star_\hbar b \ \sim \ a.b + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\hbar}{2i}\right)^k \omega^{i_1 j_1} \ldots \omega^{i_k j_k} \ \partial^k_{i_1 \ldots i_k} a \ \partial^k_{j_1 \ldots j_k} b$$

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Theorem

On $\mathbb{A}_\hbar \;:=\; C^\infty(\mathbb{R}^{2n})[[\hbar]]$, $\quad \mathbb{A}_\hbar \times \mathbb{A}_\hbar \to \mathbb{A}_\hbar : (a,b) \mapsto a \star_\hbar b$
is associative.

Did we know all this already??

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Did we know all this already??
YES!
Theorem[Weyl - von Neumann (1931)]
Canonical Schrödinger quantization (Weyl ordered):

$$\mathrm{Polynomials}(\mathbb{R}^{2n}) \longrightarrow \mathcal{L}(L^2(\mathbb{R}^n)) : a \mapsto \mathrm{Op}_\hbar(a)$$

$$\mathrm{Op}_\hbar(q^j)\varphi(q) = q^j\varphi(q) \qquad \mathrm{Op}_\hbar(p)\varphi(q) = i\hbar\partial_{q^j}\varphi(q)$$

Then

$$\mathrm{Op}_\hbar(a) \circ \mathrm{Op}_\hbar(b) = \mathrm{Op}_\hbar(a \star_\hbar b) \, .$$

Definition A **Poisson manifold** is a smooth manifold M endowed with a skewsymmetric bi-vector field w such that the associated bracket on $C^\infty(M)$:

$$\{f,g\} \;:=\; w^{ij}\partial_{x^i}f\,\partial_{x^j}g$$

satisfies $\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0$.
Examples:

- canonical phase space: $(T^\star(N),\omega_{\text{Liouville}})$
- $\mathfrak{g}$ = Lie algebra, dual: $M \;=\; \mathfrak{g}^\star$ with

$$\{f,g\}(x) \;:=<\; x\,,\,[\mathrm{d}f_x\,,\,\mathrm{d}g_x]> \qquad ((\mathfrak{g}^\star)^\star=\mathfrak{g})$$

Definition [Bayen,Flato,Fronsdal, Lichnerowicz, Sternheimer (1977)]
A **star-product** (or **deformation quantization**) on a Poisson manifold (M, w) is an associative $\mathbb{C}[[\hbar]]$-bilinear product law on $C^\infty(M)[[\hbar]] \ =: \ \mathbb{A}_\hbar$

$$\mathbb{A}_\hbar \times \mathbb{A}_\hbar \to \mathbb{A}_\hbar : (a, b) \mapsto a \star_\hbar b$$

such that
(i) $a \star_\hbar b \ = \ a.b + \sum_{k=1}^\infty \hbar^k C_k(a, b)$
(ii) $C_1(a, b) - C_1(b, a) \ = \ i\,\{a, b\}$
(iii) $C_k =$ bi-differential operator on $C^\infty(M)$ vanishing on constants.

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Remark: the proof uses (highly sophisticated) flux type methods.

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