On Drinfel'd twists and their use in non-commutative geometry

Pierre Bieliavsky

FFP14, Marseille 2014

July 17, 2014

Weights on Lie groups

Definition G : connected real Lie group G with Lie algebra $\mathfrak{g}$.
$\mu \in C^\infty(G, \mathbb{R}_0^+)$ is a **weight** if
(i) $\forall X \in \mathcal{U}(\mathfrak{g}), \exists C_L, C_R > 0$: $|\widetilde{X}.\mu| \leq C_L\, \mu$ and $|X^\star.\mu| \leq C_R\, \mu$.
(ii) $\exists L, R \in \mathbb{N}$ and $C > 0$: $\forall g, h \in G$, $\mu(gh) \leq C\, \mu^L(g)\, \mu^R(h)$.

Weights on Lie groups

Definition G : connected real Lie group G with Lie algebra $\mathfrak{g}$.
$\mu \in C^\infty(G, \mathbb{R}_0^+)$ is a **weight** if
(i) $\forall X \in \mathcal{U}(\mathfrak{g}), \exists C_L, C_R > 0$: $|\widetilde{X}.\mu| \leq C_L\,\mu$ and $|X^\star.\mu| \leq C_R\,\mu$.
(ii) $\exists L, R \in \mathbb{N}$ and $C > 0$: $\forall g, h \in G$, $\mu(gh) \leq C\,\mu^L(g)\,\mu^R(h)$.
Fréchet space $(\mathcal{E}\,,\,\{\|.\|_j\}_{j\in\mathbb{N}}) \rightsquigarrow \{\mu_j\}_{j\in\mathbb{N}}$: weights.

Weights on Lie groups

Definition G : connected real Lie group G with Lie algebra $\mathfrak{g}$.
$\mu \in C^\infty(G, \mathbb{R}_0^+)$ is a **weight** if
(i) $\forall X \in \mathcal{U}(\mathfrak{g}), \exists C_L, C_R > 0$: $|\widetilde{X}.\mu| \leq C_L\, \mu$ and $|X^\star.\mu| \leq C_R\, \mu$.
(ii) $\exists L, R \in \mathbb{N}$ and $C > 0$: $\forall g, h \in G$, $\mu(gh) \leq C\, \mu^L(g)\, \mu^R(h)$.
Fréchet space $(\mathcal{E}\,,\, \{\|.\|_j\}_{j\in\mathbb{N}}) \rightsquigarrow \{\mu_j\}_{j\in\mathbb{N}}$: weights.
$\mathcal{B}^{\{\mu_j\}}(G, \mathcal{E}) :=$

$$\left\{ F \in C^\infty(G, \mathcal{E}) \;:\; \forall X \in \mathcal{U}(\mathfrak{g}),\, \forall j \in \mathbb{N},\, \exists C > 0 \;:\; \|\widetilde{X} F\|_j \leq C\, \mu_j \right\} .$$

Weights on Lie groups

Definition G : connected real Lie group G with Lie algebra $\mathfrak{g}$.
$\mu \in C^\infty(G, \mathbb{R}_0^+)$ is a **weight** if
(i) $\forall X \in \mathcal{U}(\mathfrak{g}), \exists C_L, C_R > 0$: $|\widetilde{X}.\mu| \leq C_L\,\mu$ and $|X^\star.\mu| \leq C_R\,\mu$.
(ii) $\exists L, R \in \mathbb{N}$ and $C > 0$: $\forall g, h \in G$, $\mu(gh) \leq C\,\mu^L(g)\,\mu^R(h)$.
Fréchet space $(\mathcal{E}\,,\,\{\|.\|_j\}_{j\in\mathbb{N}}) \rightsquigarrow \{\mu_j\}_{j\in\mathbb{N}}$: weights.
$\mathcal{B}^{\{\mu_j\}}(G, \mathcal{E}) :=$
$$\left\{F \in C^\infty(G, \mathcal{E})\ :\ \forall X \in \mathcal{U}(\mathfrak{g}), \forall j \in \mathbb{N}, \exists C > 0\ :\ \|\widetilde{X}F\|_j \leq C\,\mu_j\right\}.$$
Lemma (i) $\mathcal{B}^{\{\mu_j\}}(G, \mathcal{E})$ is Fréchet.
(ii) $\mu_j \succ \mu'_j\ \forall j \in \mathbb{N} \Rightarrow \mathrm{closure}_{\mathcal{B}^{\{\mu_j\}}(G,\mathcal{E})}(\mathcal{D}(G, \mathcal{E})) \supset \mathcal{B}^{\{\mu'_j\}}(G, \mathcal{E})$

Tempered Lie groups

Definition G is **tempered** if $\exists \ G \to \mathbb{R}^m$ where the multiplication and inverse operations are tempered functions.

Tempered Lie groups

Definition G is **tempered** if $\exists \; G \to \mathbb{R}^m$ where the multiplication and inverse operations are tempered functions.

Definition Let $S \in C^\infty(G, \mathbb{R})$. (G, S) is **tempered** if

$$dS : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \mathrm{d}S_x(\widetilde{X}) = (\widetilde{X}.S)(x) \right]$$

endows G with the structure of a tempered Lie group.

Tempered Lie groups

Definition G is **tempered** if $\exists\ G \to \mathbb{R}^m$ where the multiplication and inverse operations are tempered functions.

Definition Let $S \in C^\infty(G, \mathbb{R})$. (G, S) is **tempered** if

$$dS : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \mathrm{d}S_x(\widetilde{X}) = (\widetilde{X}.S)(x)\right]$$

endows G with the structure of a tempered Lie group.

Vector space decomposition: $\mathfrak{g} = \bigoplus_{n=0}^{N} V_n$

$\forall n$: basis $\{e_j^n\}_{j=1,\dots,\dim(V_n)}$ of $V_n \rightsquigarrow x_n^j := (\widetilde{e_j^n}.S)(x)$

Tempered Lie groups

Definition G is **tempered** if $\exists\ G \to \mathbb{R}^m$ where the multiplication and inverse operations are tempered functions.

Definition Let $S \in C^\infty(G, \mathbb{R})$. (G, S) is **tempered** if

$$dS : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \mathrm{d}S_x(\widetilde{X}) = (\widetilde{X}.S)(x) \right]$$

endows G with the structure of a tempered Lie group.

Vector space decomposition: $\mathfrak{g} = \bigoplus_{n=0}^N V_n$

$\forall n$: basis $\{e_j^n\}_{j=1,\dots,\dim(V_n)}$ of $V_n \rightsquigarrow x_n^j := (\widetilde{e_j^n}.S)(x)$

Definition Tempered pair is **admissible**, if $\exists\, X_n \in \mathfrak{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$ with $\widetilde{X}_n e^{iS} =: \alpha_n e^{iS}$ such that

(i) $\exists\ C_n, \rho_n > 0$: $|\alpha_n| \geq C_n\left(1 + |x_n|_n^{\rho_n}\right) \quad (x_n := (x_n^j))$.

(ii) $\exists\ \mu_n \in C^\infty(G, \mathbb{R}_0^+)$ tempered:

(ii.1) $\forall\ A \in \mathrm{alg}_{\mathcal{U}(\mathfrak{g})}(\oplus_{k=0}^n V_k)$: $\left|\widetilde{A}\,\alpha_n\right| \leq C_A\,|\alpha_n|\,\mu_n$.

(ii.2) $\forall\ r \leq n$: $\frac{\partial \mu_n}{\partial x_r^j} = 0$.

Oscillating twists

Theorem (G, S): admissible tempered pair, μ: tempered weight, $\mathbf{m} \in \mathcal{B}^\mu(G)$ and $\{\mu'_j\}_{j\in\mathbb{N}} \succ \{\mu_j\}_{j\in\mathbb{N}}$: tempered weights. Then $\mathcal{D}(G,\mathcal{E}) \to \mathcal{E} : F \mapsto \underbrace{\int_G \mathbf{m}}e^{iS}\, F$ uniquely continuously extends to

$$\int_G \mathbf{m}\, e^{iS} : \mathcal{B}^{\{\mu_j\}}(G,\mathcal{E}) \to \mathcal{E}\ .$$

Oscillating twists

Theorem (G, S): admissible tempered pair, μ: tempered weight, $\mathbf{m} \in \mathcal{B}^{\mu}(G)$ and $\{\mu'_j\}_{j\in\mathbb{N}} \succ \{\mu_j\}_{j\in\mathbb{N}}$: tempered weights. Then $\mathcal{D}(G, \mathcal{E}) \to \mathcal{E} : F \mapsto \underbrace{\int_G \mathbf{m} e^{iS}}\, F$ uniquely continuously extends to

$$\int_G \mathbf{m}\, e^{iS} : \mathcal{B}^{\{\mu_j\}}(G, \mathcal{E}) \to \mathcal{E} \; .$$

$(\mathcal{A}, \{\|.\|_j\}_{j\in\mathbb{N}})$: Fréchet algebra.

$\mathcal{R} \otimes \mathcal{R} : C^\infty(G, \mathcal{A}) \times C^\infty(G, \mathcal{A}) \to C^\infty\big(G \times G, C^\infty(G, \mathcal{A})\big) :$

$(F, F') \longmapsto \Big[(x, y) \mapsto \big[g \mapsto F(gx)F'(gy)\big]\Big] \; .$

Oscillating twists

Theorem (G, S): admissible tempered pair, μ: tempered weight, $\mathbf{m} \in \mathcal{B}^{\mu}(G)$ and $\{\mu'_j\}_{j\in\mathbb{N}} \succ \{\mu_j\}_{j\in\mathbb{N}}$: tempered weights. Then $\mathcal{D}(G,\mathcal{E}) \to \mathcal{E} : F \mapsto \widetilde{\int_G} \mathbf{m} e^{iS} F$ uniquely continuously extends to

$$\widetilde{\int_G} \mathbf{m}\, e^{iS} : \mathcal{B}^{\{\mu_j\}}(G,\mathcal{E}) \to \mathcal{E} \,.$$

$(\mathcal{A}, \{\|.\|_j\}_{j\in\mathbb{N}})$: Fréchet algebra.

$$\mathcal{R}\otimes\mathcal{R} : C^\infty(G,\mathcal{A}) \times C^\infty(G,\mathcal{A}) \to C^\infty\big(G\times G, C^\infty(G,\mathcal{A})\big) :$$

$$(F, F') \longmapsto \Big[(x,y) \mapsto \big[g \mapsto F(gx)F'(gy)\big]\Big] \,.$$

Theorem $(G\times G, S)$: admissible. $\mathbf{m} \in \mathcal{B}^{\mu}(G\times G, \mathbb{C})$. Set

$$\star_S := \Big[(F,F') \mapsto \widetilde{\int_{G\times G}} \mathbf{m} e^{iS} \circ \mathcal{R}\otimes\mathcal{R}\,(F,F')\Big] \,.$$ Then:

$$\star_S : \mathcal{B}^{\{\mu_j\}}(G,\mathcal{A}) \times \mathcal{B}^{\{\mu'_j\}}(G,\mathcal{A}) \to \mathcal{B}^{\{\mu_j^{L_j}\mu_j'^{L'_j}\}_{j\in\mathbb{N}}}(G,\mathcal{A})$$

is a jointly continuous bilinear map.

Oscillating twists

Definition $\star_S$ is **weakly associative** if associative on $\mathcal{D}(G, \mathcal{A}) \subset \mathcal{B}^1(G, \mathcal{A})$.

Oscillating twists

Definition $\star_S$ is **weakly associative** if
associative on $\mathcal{D}(G,\mathcal{A}) \subset \mathcal{B}^1(G,\mathcal{A})$.
Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})$.

Oscillating twists

Definition $\star_S$ is **weakly associative** if
associative on $\mathcal{D}(G, \mathcal{A}) \subset \mathcal{B}^1(G, \mathcal{A})$.
Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G, \mathcal{A})$.
$\mathbb{A} = C^\star$-algebra

Oscillating twists

Definition $\star_S$ is **weakly associative** if
associative on $\mathcal{D}(G,\mathcal{A}) \subset \mathcal{B}^1(G,\mathcal{A})$.
Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})$.
$\mathbb{A} = C^\star$-algebra ,$\alpha : G \times \mathbb{A} \to \mathbb{A}$ isometrical, strongly-C^0-action

Oscillating twists

Definition $\star_S$ is **weakly associative** if associative on $\mathcal{D}(G,\mathcal{A}) \subset \mathcal{B}^1(G,\mathcal{A})$.

Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})$.

$\mathbb{A} = C^\star$-algebra ,$\alpha : G \times \mathbb{A} \to \mathbb{A}$ isometrical, strongly-C^0-action

$\mathcal{A} := \mathbb{A}^\infty = C^\infty$- vectors of α in $\mathbb{A}$ = (Fréchet) sub-algebra of $\mathbb{A}$

Oscillating twists

Definition $\star_S$ is **weakly associative** if
associative on $\mathcal{D}(G, \mathcal{A}) \subset \mathcal{B}^1(G, \mathcal{A})$.
Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G, \mathcal{A})$.
$\mathbb{A} =$ $C^\star$-algebra ,$\alpha : G \times \mathbb{A} \to \mathbb{A}$ isometrical, strongly-C^0-action
$\mathcal{A} := \mathbb{A}^\infty = C^\infty$- vectors of α in $\mathbb{A}$ = (Fréchet) sub-algebra of $\mathbb{A}$
$\mathcal{A} \to \mathcal{B}^1(G, \mathcal{A}) : a \mapsto \alpha(a) := [g \mapsto \alpha_g(a)]$ continuous injection

Oscillating twists

Definition $\star_S$ is **weakly associative** if associative on $\mathcal{D}(G,\mathcal{A}) \subset \mathcal{B}^1(G,\mathcal{A})$.

Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})$.

$\mathbb{A} = C^\star$-algebra ,$\alpha : G \times \mathbb{A} \to \mathbb{A}$ isometrical, strongly-C^0-action

$\mathcal{A} := \mathbb{A}^\infty = C^\infty$- vectors of α in $\mathbb{A}$ = (Fréchet) sub-algebra of $\mathbb{A}$

$\mathcal{A} \to \mathcal{B}^1(G,\mathcal{A}) : a \mapsto \alpha(a) := [g \mapsto \alpha_g(a)]$ continuous injection

Corollary Set

$$a \star_S^{\mathcal{A}} b := (\alpha(a) \star_S \alpha(b))(e_G) .$$

Then $(\mathcal{A}, \star_S^{\mathcal{A}})$ is a pre-$C^\star$-algebra.

Schwartz spaces

Definition (G, S): tempered. $\alpha_X := e^{-iS}\,\widetilde{X}\,e^{iS}$.
$\mathcal{S}^S(G) := \left\{ f \in C^\infty(G) \;:\; \left|\alpha_X^n(x)\left(\widetilde{Y} f\right)(x)\right| < C \right\}$.

Schwartz spaces

Definition (G,S): tempered. $\alpha_X := e^{-iS}\,\widetilde{X}\,e^{iS}$.
$\mathcal{S}^S(G) := \{f \in C^\infty(G) \;:\; |\alpha_X^n(x)\,(\widetilde{Y}\,f)(x)| < C\}$.
Definition $(G \times G, S)$: admissible. $\mathcal{A}$: Fréchet algebra.
$\mathcal{S}^S(G,\mathcal{A}) := \Big\{ [g \in G \mapsto f(g,e)] \,,\; f \in \mathcal{S}^S(G \times G, \mathcal{A}) \Big\}$.

Schwartz spaces

Definition (G, S): tempered. $\alpha_X := e^{-iS}\,\widetilde{X}\,e^{iS}$.
$\mathcal{S}^S(G) := \{f \in C^\infty(G) \ : \ |\alpha_X^n(x)\,(\widetilde{Y}\,f)(x)| < C\}$.
Definition $(G \times G, S)$: admissible. $\mathcal{A}$: Fréchet algebra.
$\mathcal{S}^S(G, \mathcal{A}) := \left\{[g \in G \mapsto f(g, e)]\,,\ f \in \mathcal{S}^S(G \times G, \mathcal{A})\right\}$.
Proposition $(G \times G, S)$: admissible. Then:
$\star_S : \mathcal{B}^{\{\mu_j\}}(G, \mathcal{A}) \times \mathcal{S}^S(G, \mathcal{A}) \to \mathcal{S}^S(G, \mathcal{A})$ is jointly continuous

Schwartz spaces

Definition (G, S): tempered. $\alpha_X := e^{-iS}\,\widetilde{X}\,e^{iS}$.
$\mathcal{S}^S(G) := \left\{ f \in C^\infty(G)\ :\ \left|\alpha_X^n(x)\,(\widetilde{Y}\,f)(x)\right| < C \right\}$.
Definition $(G \times G, S)$: admissible. $\mathcal{A}$: Fréchet algebra.
$\mathcal{S}^S(G,\mathcal{A}) := \left\{ \left[g \in G \mapsto f(g,e) \right],\ \ f \in \mathcal{S}^S(G \times G, \mathcal{A}) \right\}$.
Proposition $(G \times G, S)$: admissible. Then:
$\star_S : \mathcal{B}^{\{\mu_j\}}(G,\mathcal{A}) \times \mathcal{S}^S(G,\mathcal{A}) \to \mathcal{S}^S(G,\mathcal{A})$ is jointly continuous

Oscillating twists

Proposition (G,S): admissible. $\mathbf{A}\in\mathcal{B}^\mu(G)$ (μ tempered). Set:
$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)}\,\mathbf{A}(x)\,R^\star_x\,\mathrm{d}x\,,\ \mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)}\,\mathbf{A}(x)\,L^\star_{x^{-1}}\,\mathrm{d}x\,.$

Oscillating twists

Proposition (G, S): admissible. $\mathbf{A} \in \mathcal{B}^{\mu}(G)$ (μ tempered). Set:
$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)}\, \mathbf{A}(x)\, R^{\star}_{x}\, \mathrm{d}x\,,\ \mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)}\, \mathbf{A}(x)\, L^{\star}_{x^{-1}}\, \mathrm{d}x\ .$
Then: $\mathcal{L}_{\mathbf{A}}\,,\ \mathcal{R}_{\mathbf{A}} : \mathcal{S}^{S}(G, \mathcal{A}) \to \mathcal{S}^{S}(G, \mathcal{A})$ are continuous linear.

Oscillating twists

Proposition (G,S): admissible. $\mathbf{A} \in \mathcal{B}^{\mu}(G)$ (μ tempered). Set:

$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)}\, \mathbf{A}(x)\, R^{\star}_{x}\, \mathrm{d}x\,,\ \mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)}\, \mathbf{A}(x)\, L^{\star}_{x^{-1}}\, \mathrm{d}x\ .$

Then: $\mathcal{L}_{\mathbf{A}}\,,\ \mathcal{R}_{\mathbf{A}} : \mathcal{S}^{S}(G,\mathcal{A}) \to \mathcal{S}^{S}(G,\mathcal{A})$ are continuous linear.

Definition μ: tempered weight on $G \times G$.

$$\mathcal{B}^{\mu}_{LR}(G\times G) := \{F \in C^{\infty}(G\times G) : \left|(A^{\star}\otimes 1)(1\otimes \tilde{B}).F\right| < C\mu\}\ .$$

Oscillating twists

Proposition (G, S): admissible. $\mathbf{A} \in \mathcal{B}^\mu(G)$ (μ tempered). Set:
$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) R^\star_x \, \mathrm{d}x$, $\mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) L^\star_{x^{-1}} \, \mathrm{d}x$.
Then: $\mathcal{L}_{\mathbf{A}}$, $\mathcal{R}_{\mathbf{A}} : \mathcal{S}^S(G, \mathcal{A}) \to \mathcal{S}^S(G, \mathcal{A})$ are continuous linear.

Definition μ: tempered weight on $G \times G$.

$$\mathcal{B}^\mu_{LR}(G \times G) := \{F \in C^\infty(G \times G) : \left|(A^\star \otimes 1)(1 \otimes \tilde{B}).F\right| < C\mu\} .$$

Consider $\mathbf{A}_j \in \mathcal{B}^{\mu_j}_{LR}(G \times G)$ $(j = 1, 2)$ (μ_j tempered).
$\rightsquigarrow K_j := \mathbf{A}_j \, e^{iS}$ and $\mathcal{L}_{\mathbf{A}_j}$, $\mathcal{R}_{\mathbf{A}_j} : \mathcal{S}(G \times G) \to \mathcal{S}(G \times G)$.
Set $\star := m_0 \circ \mathcal{L}_{\mathbf{A}_1} \circ \mathcal{R}_{\mathbf{A}_2} : \mathcal{S}(G) \hat{\otimes} \mathcal{S}(G) \longrightarrow \mathcal{S}(G)$.

Oscillating twists

Proposition (G, S): admissible. $\mathbf{A} \in \mathcal{B}^{\mu}(G)$ (μ tempered). Set:
$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) R^{\star}_x \,\mathrm{d}x$, $\mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) L^{\star}_{x^{-1}} \,\mathrm{d}x$.
Then: $\mathcal{L}_{\mathbf{A}}$, $\mathcal{R}_{\mathbf{A}} : \mathcal{S}^S(G, \mathcal{A}) \to \mathcal{S}^S(G, \mathcal{A})$ are continuous linear.
Definition μ: tempered weight on $G \times G$.
$\mathcal{B}^{\mu}_{LR}(G \times G) := \{F \in C^{\infty}(G \times G) : \left|(A^{\star} \otimes 1)(1 \otimes \tilde{B}).F\right| < C\mu\}$.
Consider $\mathbf{A}_j \in \mathcal{B}^{\mu_j}_{LR}(G \times G)$ $(j = 1, 2)$ (μ_j tempered).
$\rightsquigarrow K_j := \mathbf{A}_j e^{iS}$ and $\mathcal{L}_{\mathbf{A}_j}$, $\mathcal{R}_{\mathbf{A}_j} : \mathcal{S}(G \times G) \to \mathcal{S}(G \times G)$.
Set $\star := m_0 \circ \mathcal{L}_{\mathbf{A}_1} \circ \mathcal{R}_{\mathbf{A}_2} : \mathcal{S}(G) \hat{\otimes} \mathcal{S}(G) \longrightarrow \mathcal{S}(G)$.
Proposition $(G \times G, S)$: admissible. $\mu' \succ \mu$. Then
$\mathcal{B}^{\mu}_{LR}(G \times G) \times \mathcal{S}(G \times G) \to \mathcal{S}(G \times G) : (F, \Phi) \mapsto F \star \otimes \star \Phi$
($\mathcal{B}^{\mu}_{LR}(G \times G) \subset \mathrm{closure}_{\mathcal{B}^{\mu'}_{LR}(G \times G)}(\mathcal{D}(G \times G))$)

Oscillating twists

Proposition (G,S): admissible. $\mathbf{A}\in\mathcal{B}^\mu(G)$ (μ tempered). Set:
$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)}\,\mathbf{A}(x)\,R^\star_x\,\mathrm{d}x\,,\ \mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)}\,\mathbf{A}(x)\,L^\star_{x^{-1}}\,\mathrm{d}x\,.$
Then: $\mathcal{L}_{\mathbf{A}}\,,\,\mathcal{R}_{\mathbf{A}} : \mathcal{S}^S(G,\mathcal{A})\to\mathcal{S}^S(G,\mathcal{A})$ are continuous linear.

Definition μ: tempered weight on $G\times G$.
$\mathcal{B}^\mu_{LR}(G\times G) := \{F\in C^\infty(G\times G) : \left|(A^\star\otimes 1)(1\otimes\tilde{B}).F\right| < C\mu\}\,.$

Consider $\mathbf{A}_j \in \mathcal{B}^{\mu_j}_{LR}(G\times G)$ $\quad(j=1,2)$ (μ_j tempered).
$\rightsquigarrow K_j := \mathbf{A}_j\,e^{iS}$ and $\mathcal{L}_{\mathbf{A}_j}\,,\,\mathcal{R}_{\mathbf{A}_j} : \mathcal{S}(G\times G)\to\mathcal{S}(G\times G)\,.$
Set $\star := m_0\circ\mathcal{L}_{\mathbf{A}_1}\circ\mathcal{R}_{\mathbf{A}_2} : \mathcal{S}(G)\hat{\otimes}\mathcal{S}(G)\longrightarrow\mathcal{S}(G)$.

Proposition $(G\times G,S)$: admissible. $\mu'\succ\mu$. Then
$\mathcal{B}^\mu_{LR}(G\times G)\times\mathcal{S}(G\times G)\to\mathcal{S}(G\times G) : (F,\Phi)\mapsto F\star\!\otimes\!\star\,\Phi$
$\left(\ \mathcal{B}^\mu_{LR}(G\times G)\subset \mathrm{closure}_{\mathcal{B}^{\mu'}_{LR}(G\times G)}(\mathcal{D}(G\times G))\ \right)$

Lemma $\forall z\in G\times G$: $ev_z\circ\star\!\otimes\!\star : \mathcal{D}(G\times G)\times\mathcal{D}(G\times G)\to\mathbb{C}$
uniquely extends as a separately continuous bilinear map:
$\mathcal{B}^\mu_{LR}(G\times G)\times\mathcal{B}^\mu_{LR}(G\times G)\to\mathbb{C}\,.$

Oscillating twists

Proposition (G, S): admissible. $\mathbf{A} \in \mathcal{B}^{\mu}(G)$ (μ tempered). Set:

$\mathcal{L}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) R^{\star}_x \,\mathrm{d}x \,,\; \mathcal{R}_{\mathbf{A}} := \int_G e^{iS(x)} \mathbf{A}(x) L^{\star}_{x^{-1}} \,\mathrm{d}x$.

Then: $\mathcal{L}_{\mathbf{A}} \,,\, \mathcal{R}_{\mathbf{A}} : \mathcal{S}^S(G, \mathcal{A}) \to \mathcal{S}^S(G, \mathcal{A})$ are continuous linear.

Definition μ: tempered weight on $G \times G$.

$\mathcal{B}^{\mu}_{LR}(G \times G) := \{F \in C^{\infty}(G \times G) : \left|(A^{\star} \otimes 1)(1 \otimes \tilde{B}).F\right| < C\mu\}$.

Consider $\mathbf{A}_j \in \mathcal{B}^{\mu_j}_{LR}(G \times G)$ $(j = 1, 2)$ (μ_j tempered).

$\rightsquigarrow K_j := \mathbf{A}_j e^{iS}$ and $\mathcal{L}_{\mathbf{A}_j} \,,\, \mathcal{R}_{\mathbf{A}_j} : \mathcal{S}(G \times G) \to \mathcal{S}(G \times G)$.

Set $\star := m_0 \circ \mathcal{L}_{\mathbf{A}_1} \circ \mathcal{R}_{\mathbf{A}_2} : \mathcal{S}(G) \hat{\otimes} \mathcal{S}(G) \longrightarrow \mathcal{S}(G)$.

Proposition $(G \times G, S)$: admissible. $\mu' \succ \mu$. Then

$\mathcal{B}^{\mu}_{LR}(G \times G) \times \mathcal{S}(G \times G) \to \mathcal{S}(G \times G) : (F, \Phi) \mapsto F \star\otimes\star \Phi$

$\left(\mathcal{B}^{\mu}_{LR}(G \times G) \subset \mathrm{closure}_{\mathcal{B}^{\mu'}_{LR}(G \times G)}(\mathcal{D}(G \times G)) \right)$

Lemma $\forall z \in G \times G$: $ev_z \circ \star\otimes\star : \mathcal{D}(G \times G) \times \mathcal{D}(G \times G) \to \mathbb{C}$ uniquely extends as a separately continuous bilinear map:

$\mathcal{B}^{\mu}_{LR}(G \times G) \times \mathcal{B}^{\mu}_{LR}(G \times G) \to \mathbb{C}$.

Schwartz co-algebras

Lemma Co-product: $\Delta : \mathcal{S}(G) \to \mathcal{M}(\mathcal{S}(G)\hat{\otimes}\mathcal{S}(G)) : \mathcal{S}(G)$ is a (commutative) multiplier Fréchet-Hopf algebra.

Remark $\Delta : \mathcal{S}^S(G) \to \mathcal{B}^{\mu}_{LR}(G \times G) \subset \mathcal{M}_{\star}(\mathcal{S}(G)\hat{\otimes}\mathcal{S}(G))$.

Proposition The condition:

$$\int_{G\times G} \Delta^R(\xi\eta)\, K_1(x\xi, y\eta)\, K_2(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \;=\; \delta_{(e,e)}(x,y) \qquad \text{(INV)}$$

$(\mathrm{d}(x\xi) \;=:\; \Delta^R(\xi)\, \mathrm{d}x)$

implies $\Delta(\varphi \star \psi) \;=\; \Delta(\varphi) \star\!\otimes\!\star\, \Delta(\psi)$ on $\mathcal{S}(G \times G)$.

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator: $W \;:=\; (1\otimes m_0)(\Delta\otimes 1) \text{ on } \mathcal{S}(G\times G)\;.$

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator: $W := (1 \otimes m_0)(\Delta \otimes 1) \text{ on } \mathcal{S}(G \times G)$.
Properties: $W : L^2_r(G)\hat{\otimes}L^2_r(G) \to L^2_r(G)\hat{\otimes}L^2_r(G)$ unitary
$W_{12}W_{13}W_{23} = W_{23}W_{12}$ [*pentagon equation*]

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator: $W := (1 \otimes m_0)(\Delta \otimes 1) \text{ on } \mathcal{S}(G \times G)$.
Properties: $W : L^2_r(G)\hat{\otimes}L^2_r(G) \to L^2_r(G)\hat{\otimes}L^2_r(G)$ unitary
$W_{12}W_{13}W_{23} = W_{23}W_{12}$ [*pentagon equation*]
Definition $(G \times G, S)$ admissible.
Deformed Kac-Takesaki operator:
$W_\star(\varphi_1 \otimes \varphi_2)(x, y) := \lim_{\psi_n \to 1} (\Delta(\varphi_1) \star (\psi_n \otimes \varphi_2)(x, y))$
(limit in $\mathcal{B}^\mu_{LR}(G \times G)$).

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator: $W := (1 \otimes m_0)(\Delta \otimes 1)$ on $\mathcal{S}(G \times G)$.

Properties: $W : L^2_r(G)\hat{\otimes} L^2_r(G) \to L^2_r(G)\hat{\otimes} L^2_r(G)$ unitary

$W_{12}W_{13}W_{23} = W_{23}W_{12}$ [*pentagon equation*]

Definition $(G \times G, S)$ admissible.

Deformed Kac-Takesaki operator:

$W_\star(\varphi_1 \otimes \varphi_2)(x,y) := \lim_{\psi_n \to 1} (\Delta(\varphi_1) \star (\psi_n \otimes \varphi_2)(x,y))$

(limit in $\mathcal{B}^\mu_{LR}(G \times G)$).

Theorem Assume:

- $\star_j := \mu_0 \circ \mathcal{L}_{\mathbf{A}_j}$: weakly associative $(j = 1, 2)$
- $\star_j$ strongly closed [Connes-Flato]
- Condition (INV).

Set $< \varphi_1, \varphi_2 >_\star := \int_G (\overline{\varphi_1} \star \varphi_2)(g) \, \mathrm{d}_r g$.

Then (i) $(\mathcal{S}(G) \, , < , >_\star)$ pre-Hilbert $\rightsquigarrow \mathcal{H}_\star$

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator: $W := (1 \otimes m_0)(\Delta \otimes 1)$ on $\mathcal{S}(G \times G)$.

Properties: $W : L^2_r(G)\hat{\otimes}L^2_r(G) \to L^2_r(G)\hat{\otimes}L^2_r(G)$ unitary

$W_{12}W_{13}W_{23} = W_{23}W_{12}$ [*pentagon equation*]

Definition $(G \times G, S)$ admissible.

Deformed Kac-Takesaki operator:

$W_\star(\varphi_1 \otimes \varphi_2)(x,y) := \lim_{\psi_n \to 1} \left(\Delta(\varphi_1) \star (\psi_n \otimes \varphi_2)(x,y)\right)$

(limit in $\mathcal{B}^\mu_{LR}(G \times G)$).

Theorem Assume:

- $\star_j := \mu_0 \circ \mathcal{L}_{\mathbf{A}_j}$: weakly associative ($j = 1, 2$)
- $\star_j$ strongly closed [Connes-Flato]
- Condition (INV).

Set $< \varphi_1, \varphi_2 >_\star := \int_G (\overline{\varphi_1} \star \varphi_2)(g)\, \mathrm{d}_r g$.

Then (i) $(\mathcal{S}(G), <, >_\star)$ pre-Hilbert $\rightsquigarrow \mathcal{H}_\star$

(ii) $W_\star$ is a multiplicative unitary on $\mathcal{H}_\star\hat{\otimes}\mathcal{H}_\star$.

Oscillating twists for Kahlerian Lie groups I: old result

Definition Let $N \in \mathbb{N}_0$.

$\mathbf{\Theta}_N := \{\tau \in C^\infty(\mathbb{R}^N, \mathbb{C}) \mid \exp(\pm\tau) \in \mathcal{O}_C(\mathbb{R}^N, \mathbb{C})\}$

Theorem $\mathbb{B}$: negatively curved rank N Kahlerian Lie group. $\exists$

- $\hat{S} \in C^\infty(\mathbb{B} \times \mathbb{B} \times \mathbb{B}, \mathbb{R})^{\mathbb{B}}$
- $\mathbf{\Theta}_N \to \mathcal{B}^\mu(\mathbb{B} \times \mathbb{B} \times \mathbb{B})^{\mathbb{B}} \,:\, \tau \mapsto \hat{\mathbf{A}}_\tau$
- a left-invariant function sub-space $\mathcal{D}(\mathbb{B}) \subset \mathcal{A}_\tau \subset C_0^\infty(\mathbb{B})$

such that:

(i) the formula ($\theta \in \mathbb{R}_0$):

$$\varphi_1 \star_{\theta,\tau} \varphi_2(x) := \frac{1}{\theta^{\dim \mathbb{B}}} \int_{\mathbb{B}\times\mathbb{B}} \hat{\mathbf{A}}_\tau(x,y,z) e^{\frac{2i}{\theta}\hat{S}(x,y,z)} \varphi_1(y)\varphi_2(z)\mathrm{d}y\mathrm{d}z$$

extends from $\mathcal{D}(\mathbb{B}) \times \mathcal{D}(\mathbb{B})$ to a left-invariant associative algebra structure on $\mathcal{A}_\tau$.

(ii) $\varphi_1 \star_{\theta,\tau} \varphi_2 \;\sim\; \varphi_1\varphi_2 + \frac{\theta}{2i}\{\varphi_1\,,\,\varphi_2\} + \ldots.$ (formal star-product).

Oscillating twists for Kahlerian Lie groups II: admissibility and pentagon equation

Theorem Define: $S(x,y) := \hat{S}(x,y,e)$, $\mathbf{A}_\tau(x,y) := \hat{\mathbf{A}}_\tau(x,y,e)$

$$K_\tau := \frac{1}{\theta^{\dim \mathbb{B}}} \mathbf{A}_\tau e^{\frac{2i}{\theta} S} .$$

Then

(i) $(\mathbb{B} \times \mathbb{B}, S)$ is an admissible tempered pair.

(ii) $\tau \in \mathcal{B}^{\mu_0}(\mathbb{R}^N) \Rightarrow \mathbf{A}_\tau \in \mathcal{B}^{\mu}_{LR}(\mathbb{B} \times \mathbb{B})$.

(iii) the pair $(K_\tau, K_{-\overline{\tau}})$ satisfies condition (INV).

Oscillating twists for Kahlerian Lie groups II: admissibility and pentagon equation

Theorem Define: $S(x, y) := \hat{S}(x, y, e)$, $\mathbf{A}_\tau(x, y) := \hat{\mathbf{A}}_\tau(x, y, e)$

$$K_\tau := \frac{1}{\theta^{\dim \mathbb{B}}} \mathbf{A}_\tau e^{\frac{2i}{\theta} S} \ .$$

Then

(i) $(\mathbb{B} \times \mathbb{B}, S)$ is an admissible tempered pair.

(ii) $\tau \in \mathcal{B}^{\mu_0}(\mathbb{R}^N) \ \Rightarrow \ \mathbf{A}_\tau \in \mathcal{B}^{\mu}_{LR}(\mathbb{B} \times \mathbb{B})$.

(iii) the pair $(K_\tau, K_{-\overline{\tau}})$ satisfies condition (INV).

Corollary

- Explicit Drinfel'd twists for $C^\star$ or Fréchet $\mathbb{B}$-module-algebras
- $\star^\tau_\theta \ := \ \mu_0 \circ \mathcal{L}_{\mathbf{A}_\tau} \circ \mathcal{R}_{\mathbf{A}_{-\overline{\tau}}} \ \Rightarrow \ W_{\star^\tau_\theta}$ multiplicative unitaries.

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.
Iwaswa decomposition $G = ANK$.

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.
Iwaswa decomposition $G = ANK$.
$\mathbb{B} := AN \rightsquigarrow$ dressing: $\mathbb{B} \times K \to K : (x, k) \mapsto k^x \; (kx =: x^k k^x)$

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.
Iwaswa decomposition $G = ANK$.
$\mathbb{B} := AN \rightsquigarrow$ dressing: $\mathbb{B} \times K \to K : (x, k) \mapsto k^x \; (kx =: x^k k^x)$
Embedding: $\mathcal{S}(\mathbb{B}, C^\infty(K)) \to C^\infty(G) : \varphi \mapsto [xk \mapsto \varphi(x)(k)]$

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.

Iwaswa decomposition $G = ANK$.

$\mathbb{B} := AN \rightsquigarrow$ dressing: $\mathbb{B} \times K \to K : (x, k) \mapsto k^x$ $(kx =: x^k k^x)$

Embedding: $\mathcal{S}(\mathbb{B}, C^\infty(K)) \to C^\infty(G) : \varphi \mapsto [xk \mapsto \varphi(x)(k)]$

Co-product: $\Delta\varphi(xk, x'k') := \varphi(x\, x'^k)(k^{x'}\, k')$

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.
Iwaswa decomposition $G = ANK$.
$\mathbb{B} := AN \rightsquigarrow$ dressing: $\mathbb{B} \times K \to K : (x, k) \mapsto k^x \; (kx =: x^k k^x)$
Embedding: $\mathcal{S}(\mathbb{B}, C^\infty(K)) \to C^\infty(G) : \varphi \mapsto [xk \mapsto \varphi(x)(k)]$
Co-product: $\Delta\varphi(xk, x'k') \; := \; \varphi(x\,x'^k)(k^{x'}k')$
<u>Lemma</u> $C(K)_\infty$: $C^\infty-$vectors of dressing action of $\mathbb{B}$ on $C(K)$.
Then: $\Delta : \mathcal{S}(\mathbb{B}, C(K)_\infty) \to \mathcal{B}^1_{LR}(\mathbb{B} \times \mathbb{B}, C(K \times K)_\infty)$.
Fréchet-valued oscillatory twists:

$$\mathcal{L}_{\mathbf{A}_\tau}, \mathcal{R}_{\mathbf{A}_{-\overline{\tau}}} : \mathcal{S}(\mathbb{B} \times \mathbb{B}, C(K)_\infty) \to \mathcal{S}(\mathbb{B} \times \mathbb{B}, C(K)_\infty) \; \rightsquigarrow \; \star^\tau_\theta \, .$$

Application I: Multiplicative unitaries for quantum semi-simple Lie groups

G: Hermitean type non-compact simple Lie group.
Iwaswa decomposition $G = ANK$.
$\mathbb{B} := AN \rightsquigarrow$ dressing: $\mathbb{B} \times K \to K : (x, k) \mapsto k^x \; (kx =: x^k k^x)$
Embedding: $\mathcal{S}(\mathbb{B}, C^\infty(K)) \to C^\infty(G) : \varphi \mapsto [xk \mapsto \varphi(x)(k)]$
Co-product: $\Delta\varphi(xk, x'k') := \varphi(x\, x'^k)(k^{x'} k')$
Lemma $C(K)_\infty$: $C^\infty-$vectors of dressing action of $\mathbb{B}$ on $C(K)$.
Then: $\Delta : \mathcal{S}(\mathbb{B}, C(K)_\infty) \to \mathcal{B}^1_{LR}(\mathbb{B} \times \mathbb{B}, C(K \times K)_\infty)$.
Fréchet-valued oscillatory twists:

$$\mathcal{L}_{\mathbf{A}_\tau}, \mathcal{R}_{\mathbf{A}_{-\overline{\tau}}} : \mathcal{S}(\mathbb{B} \times \mathbb{B}, C(K)_\infty) \to \mathcal{S}(\mathbb{B} \times \mathbb{B}, C(K)_\infty) \rightsquigarrow \star^\tau_\theta .$$

Proposition Set $\mathcal{H}_\star(G) := \mathcal{H}_{\theta,\tau} \otimes L^2(K)$.
Then, the $C(K)_\infty$-valued deformed Kac-Takesaki operator $W_{\theta,\tau}$ extends as a multiplicative unitary operator on $\mathcal{H}_\star(G) \otimes \mathcal{H}_\star(G)$.

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1, n) \ltimes \mathbb{R}^{1+n}$.

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1,n) \ltimes \mathbb{R}^{1+n}$. Iwasawa: $SO_0(1,n) = SO(n).AN$

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1,n) \ltimes \mathbb{R}^{1+n}$. Iwasawa: $SO_0(1,n) = SO(n).AN$
$\Rightarrow\ G\ :=\ AN \ltimes \mathbb{R}^{1,n}$ tempered Lie group.

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1,n) \ltimes \mathbb{R}^{1+n}$. Iwasawa: $SO_0(1,n) = SO(n).AN$
$\Rightarrow$ $G := AN \ltimes \mathbb{R}^{1,n}$ tempered Lie group.

Proposition (i) Every time-like direction e_0 in $\mathbb{R}^{1,n} \subset G$ is supplementary to a Frobenius sub-group $\mathbb{F}_{e_0} \subset G$.
(ii) $\mathbb{F}_{e_0}$ simply transitively acts on the *rank-one* negatively curved Kahlerian group $\mathbb{B}$ of dimension $2n$.
(iii) $\mathbb{F}_{e_0} \subset \mathtt{Aut}(\star_{\theta,\tau})$.

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1,n) \ltimes \mathbb{R}^{1+n}$. Iwasawa: $SO_0(1,n) = SO(n).AN$
$\Rightarrow \; G \; := \; AN \ltimes \mathbb{R}^{1,n}$ tempered Lie group.

Proposition (i) Every time-like direction e_0 in $\mathbb{R}^{1,n} \subset G$ is supplementary to a Frobenius sub-group $\mathbb{F}_{e_0} \; \subset \; G$.
(ii) $\mathbb{F}_{e_0}$ simply transitively acts on the *rank-one* negatively curved Kahlerian group $\mathbb{B}$ of dimension $2n$.
(iii) $\mathbb{F}_{e_0} \; \subset \; \mathtt{Aut}(\star_{\theta,\tau})$.

$\Rightarrow \mathbb{F}_{e_0}$ carries an oscillating twist (transported from the one on $\mathbb{B}$ through symplectomorphism: $\mathbb{F}_{e_0} \to \mathbb{B} : x \mapsto x.e$).

Application II: quantum Poincaré groups Minkowski modules

Poincaré: $SO_0(1,n) \ltimes \mathbb{R}^{1+n}$. Iwasawa: $SO_0(1,n) = SO(n).AN$
$\Rightarrow \ G \ := \ AN \ltimes \mathbb{R}^{1,n}$ tempered Lie group.

Proposition (i) Every time-like direction e_0 in $\mathbb{R}^{1,n} \subset G$ is supplementary to a Frobenius sub-group $\mathbb{F}_{e_0} \subset G$.
(ii) $\mathbb{F}_{e_0}$ simply transitively acts on the *rank-one* negatively curved Kahlerian group $\mathbb{B}$ of dimension $2n$.
(iii) $\mathbb{F}_{e_0} \subset \mathtt{Aut}(\star_{\theta,\tau})$.

$\Rightarrow \mathbb{F}_{e_0}$ carries an oscillating twist (transported from the one on $\mathbb{B}$ through symplectomorphism: $\mathbb{F}_{e_0} \to \mathbb{B} : x \mapsto x.e$).

$\rightsquigarrow$ Deformed Minkowski space:

Semi-classical Poisson structure ($e_0 = (1,0,...,0)$):
$\tilde{x}^1 \ := \ x^1 - x^0 \, , \, \tilde{x}^k \ := \ x^k \ (k \neq 1) \ \rightsquigarrow \ \{ \tilde{x}^0 \, , \, \tilde{x}^k \} \ = \ \tilde{x}^k \ .$