Causality and Noncommutative Geometry

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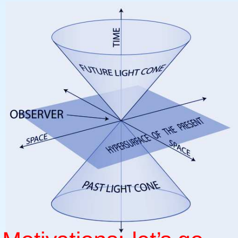

Motivations: let's go causal !

1. First motivation: sheer curiosity !

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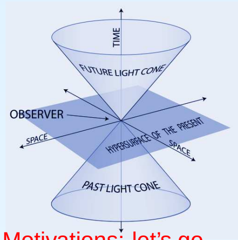
   What algebraic properties caracterize the set of causal functions, and do they make sense in a NC algebra ?



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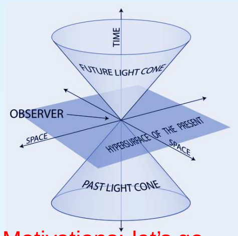

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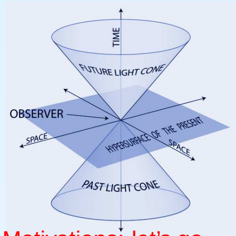

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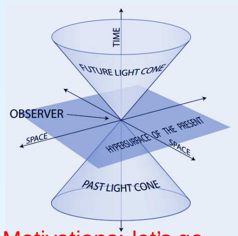

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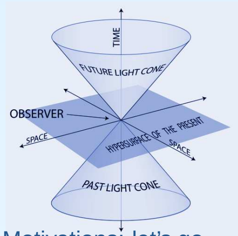
Advantage : Causality contains a lot of information
(R. Sorkin: " Geometry = order + number")



The dictionnary of Noncommutative Geometry

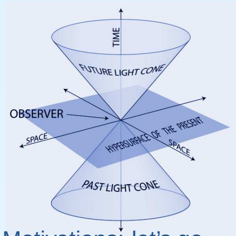
The commutative Gelfand-Naimark theorem of 1943 :

Compact spaces$+$continuous maps $\rightleftharpoons$ Abelian C^*-alg$+C^*-$morphisms

The maps are

$$X \longmapsto \mathcal{C}(X,\mathbb{C}) + \|\ \|_\infty,\ A \longmapsto \hat{A} + * - \text{weak topology}$$

and pullbacks. ($\hat{A} =$ character space $=$ pure state space)



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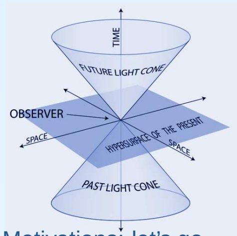
Gelfand transform: $f(x) = x(f)$

Geometry/Topology	Algebra
locally compact space	abelian C^*-algebra
compact	unital
measured space	abelian VN algebra
vector bundles	projective modules of finite type



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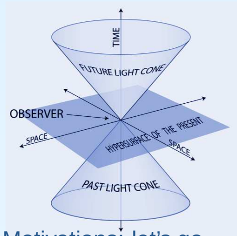

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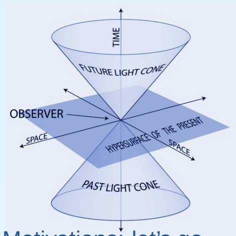
Causality $=$ partial order. What does correspond to partial ordered spaces in the table ?



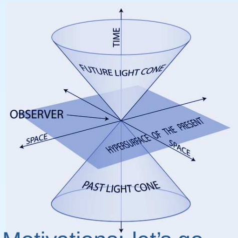
The duality result (1)

A few definitions :

1. Let $(M, \preceq)$, $(N, \leq)$ be two posets. A map $f : M \longrightarrow N$ is *isotone* iff $x \preceq y \Rightarrow f(x) \leq f(y)$.

2. Let $(M, \preceq, \tau)$ be poset $+$ a topology on it. We set $I(M) := \{f : M \longrightarrow \mathbb{R} | f$ is isotone and continuous$\}$.

3. $(M, \preceq, \tau)$ is called a *toposet* iff $x \preceq y \Leftrightarrow \forall f \in I(M), f(x) \leq f(y)$.



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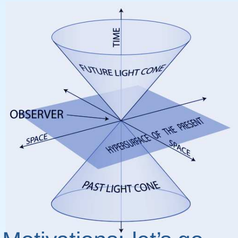

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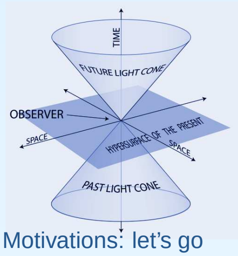
- toposet $=$ necessary condition to have a duality result.
- Physically : all observers agree on the chronology of two events exactly when they are causally connected.

Example: any globally hyperbolic manifold.



The duality result (2)

Definition Let A be a C^*-algebra with unit 1. A subset I of $\mathrm{Re}(A)$ which satisfies :

1. $\forall x \in \mathbb{R}, x.1 \in I$.
2. $\forall b, b' \in I, b + b' \in I$,
3. $\forall \phi \in I(\mathbb{R}),\ \phi(I) \subset I$,
4. $\overline{I} = I$,
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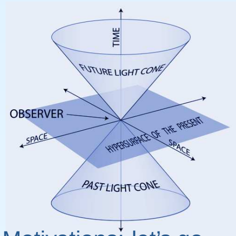
is called an isocone. The couple (I, A) is called an I^*-algebra.



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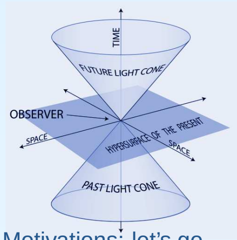
(compact toposets, cont. real isotonies) $\rightleftharpoons$ (abelian I^*-alg., I^*-morph.)



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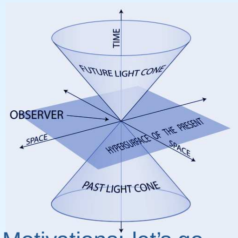
Rem 1: the formula $\phi \preceq_I \psi \Longleftrightarrow \forall a \in I, \phi(a) \leq \psi(a)$ always defines a toposet structure on $P(A)$.
Rem 2: by the duality result, $I \cap C^*(a)$ induces a toposet structure on $\sigma(a)$, for $a \in I$.



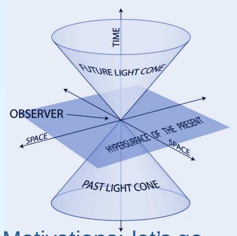
Isocones from a physical point of view

$a \in I \iff a$ is a causal observable = "unscaled clock"

(Conservative) assumptions:

1. A reading of a is in $\sigma(a)$.
2. The expected value of a in ϕ is $\phi(a)$.
3. When in (pure) state $|x\rangle$, for $x \in \sigma(a)$, a will yield the result x with certainty.

- Spacetime $\longrightarrow P =$ set of all $|x\rangle$, $x \in \sigma(a)$, $a \in I$.
- Causality : $\phi, \eta \in P$, $\phi \preceq \eta \Leftrightarrow \phi(a) \leq \eta(a)$ for every $a \in I$. (Reading of clock at $\phi \leq$ reading of clock at η on the average, for every possible clock.)



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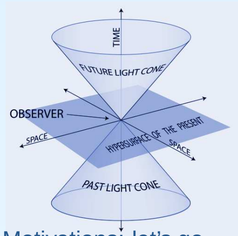
$\Longrightarrow I$ is a closed convex cone that contains the constants, is stable by isotone calculus and separates the states.



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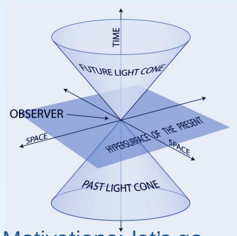
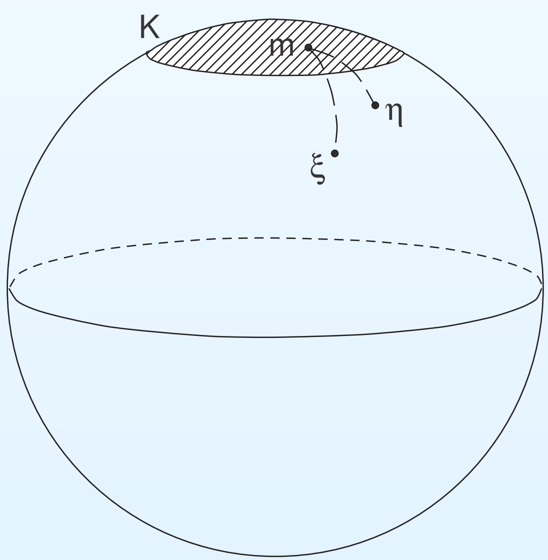
$\Longrightarrow I$ is an isocone under the Stone-Weierstrass conjecture for Jordan algebras.

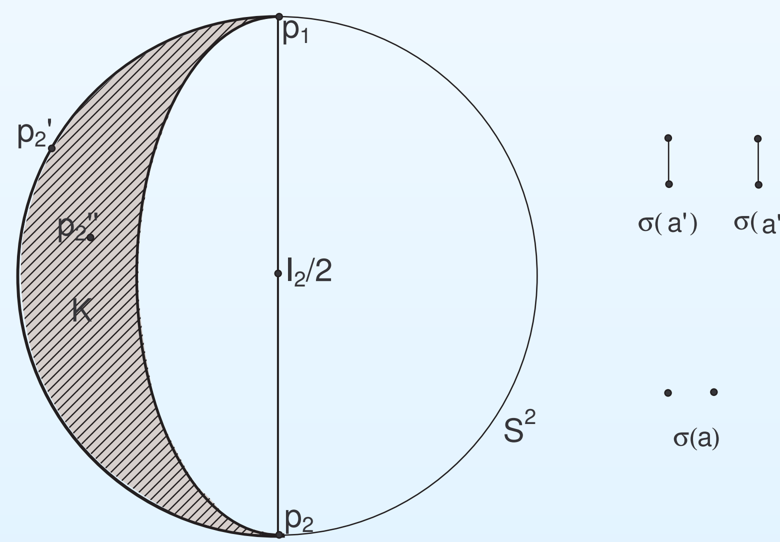


A finite dimensional example

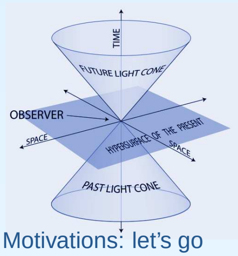
Let $A = M_2(\mathbb{C})$. Then stability under isotone calculus follows from the other axioms.
What are the orders induced by I on $P(A)$? On $\sigma(a)$, $a \in I$?
$P(A) = \mathbb{P}(\mathbb{C}^2) \simeq \mathcal{P}_1 = \{$rank 1 projections$\} \simeq S^2$.
pure states : $\phi_{[\xi]}(a) = \langle \xi, a\xi \rangle = \mathrm{Tr}(ap_\xi)$.
Let $K = I \cap \mathcal{P}$. I isocone iff $K = S^2$ or $K =$ closed, geodesically convex subset of a half-sphere, with non-empty interior.

$$\phi_{[\xi]} \leq_I \phi_{[\eta]} \Leftrightarrow \forall m \in K, d(m, p_\xi) \geq d(m, p_\eta)$$



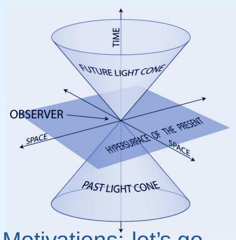


$p_2, p_2', p_2'' =$projection corresponding to largest eigenvalue of a, a', a''



An infinite dimensional example

Basic idea : an increasing function $+$ an infinitesimal perturbation is still increasing.



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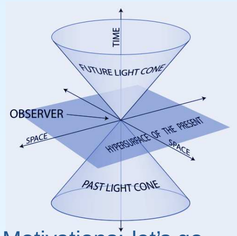
Theorem Let A be a sub-C^*-algebra of B, I be an isocone of A and K be a closed 2-sided ideal of B.
Then $(\overline{I + \text{Re } K}, A + K)$ is an I^*-algebra.



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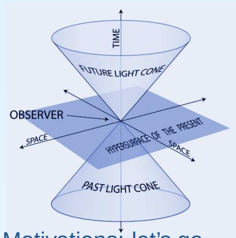
In particular, let $(M, \leq)$ be a compact toposet, $H = L^2(M)$,
$A = \{M_f : \phi \mapsto f\phi | f \in \mathcal{C}(M)\} \subset B(H)$, $I = \{M_f | f \in I(M)\}$, and
$K = \{$ compact operators $\}$. Then $I + \mathrm{Re}\, K$ is closed.
$\rightarrow (I + \mathrm{Re}\, K, A + K)$ is an example of an infinite dimensional NC I^*-algebra.
What is the order induced on $P(A + K)$?

$$(P(A+K), \preceq_I) \simeq (M, \leq) + (P^K(A), =)$$

Pure states vanishing on K = Dirac delta functions on M.
Pure states not vanishing on K = vector states $a \mapsto \langle \psi, a\psi \rangle$, $\psi \in H$ = wave functions on M : incomparable with each other.

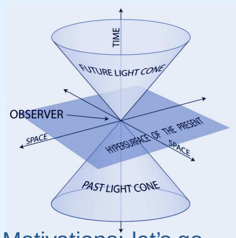


Lexicographic sums of isocones

Consider a family of posets $(M_x, \preceq_x)_{x \in P}$, P poset.

The *lexicographic sum* is $\operatorname*{Lex}_{x \in P} M_x := (\coprod_{x \in P} M_x, \preceq)$ with the ordering :

$\preceq = \preceq_x$ on M_x and $x \prec y \Rightarrow M_x \prec M_y$, $x \parallel y \Rightarrow M_x \parallel M_y$.



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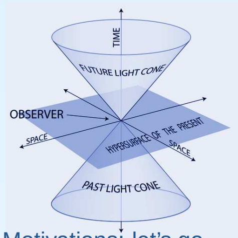
Proposition Let $(P, \preceq)$ be a finite poset and for each $x \in P$ let (I_x, A_x) be an I^*-algebra. We set $I = \bigoplus_{x \in P} I_x$, $A = \bigoplus_{x \in P} A_x$. Then

$$\operatorname{Lex}_{x \in P} I_x := \{a \in I | \forall x, y \in P, x \prec y \Rightarrow \max \sigma(a_x) \leq \min \sigma(a_y)\}$$

is an isocone of A.
Moreover

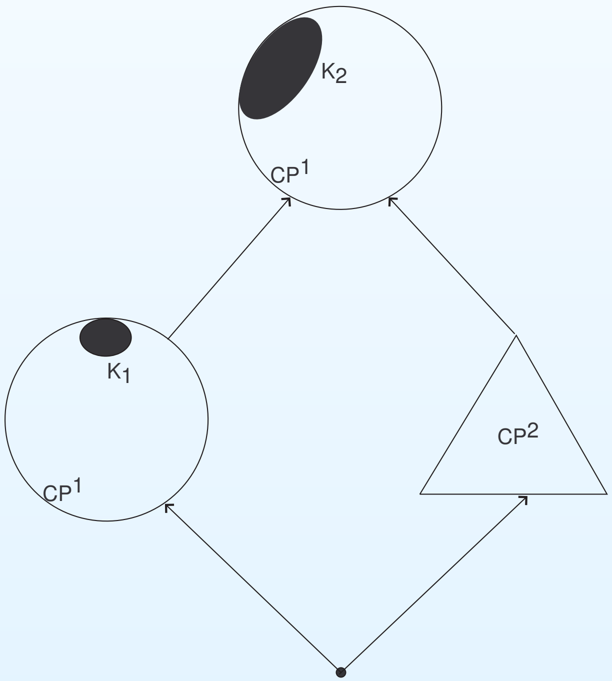
$$P(A) \approx \operatorname{Lex}_{x \in P} P(A_x)$$

as toposets, where $P(A)$ is equipped with the ordering $\preceq_J$, with $J = \operatorname{Lex}_{x \in P} I_x$, and $P(A_x)$ is equipped with $\preceq_{I_x}$.

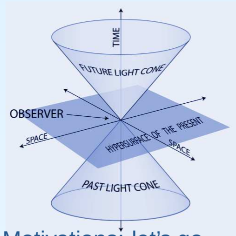


Classification result in finite dimension

Theorem Let I be an isocone in the finite-dimensional C^*-algebra $A = \bigoplus_{x \in P} M_{n_x}(\mathbb{C})$, with $P = \{1; \ldots ; k\}$, $k \in \mathbb{N}^*$, $n_x \in \mathbb{N}^*$. Then there exists a poset structure on P, and isocones $I_x \subset \mathrm{Re} M_{n_x}(\mathbb{C})$ such that $I = \mathrm{Lex}_{x \in P} I_x$.
Moreover, if $n_x \neq 2$ then $I_x = \mathrm{Re} M_{n_x}(\mathbb{C})$.



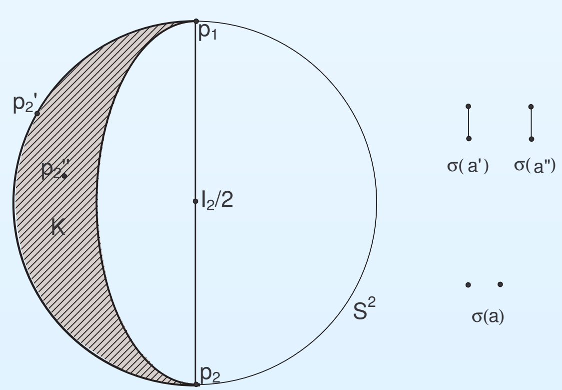
Lexicographic sum of $\mathbb{R}$, $I_{K_1} \subset \mathrm{Re} M_2(\mathbb{C})$, $\mathrm{Re} M_3(\mathbb{C})$, and $I_{K_2} \subset \mathrm{Re} M_2(\mathbb{C})$ over $P = \Diamond$.

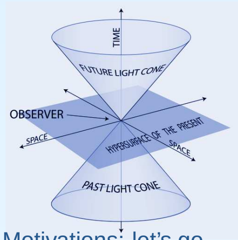


Classification: sketch of proof

Hardest part : triviality of $M_3(\mathbb{C})$.
Main tools :

- If $a \in I$ then the projection on the eigenspace for the k largest e.v. belongs to I.
- The projections in I form a lattice.
- If rank one projection p and rank two projection π do not commute, then p allows to decompose π into the sum of 2 rank one projections in I.
- Local constancy of “inner ordering” on the non-degenerate part of I.





The almost commutative case

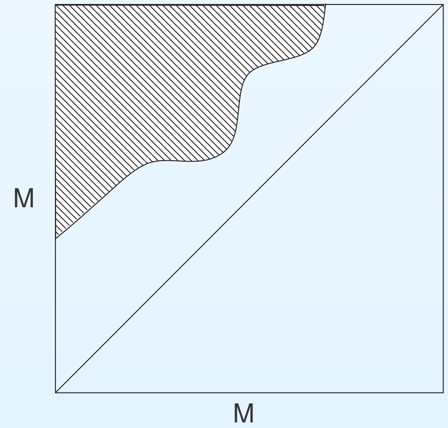
Consider $A = \mathcal{C}(M) \otimes M_n(\mathbb{C})$ for simplicity. Let $\mathcal{I}_n$ be the set of isocones of $M_n(\mathbb{C})$.

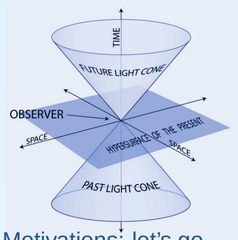
Theorem [NB-FB] Let I be an isocone of A. Then the order induced by I on $M \times \mathbb{P}(\mathbb{C}^n)$ is lexicographic: there exists an ordering $\preceq_M$ on M and for each $x \in M$ there exists $I_x \in \mathcal{I}_n$ such that :

$$(x,\xi) \preceq_I (y,\eta) \Leftrightarrow \begin{cases} x \prec_M y \\ \text{or} \\ x = y \text{ and } \xi \preceq_{I_x} \eta \end{cases}$$

Theorem [NB-FB] Assume M is metrizable. Let $\preceq$ be a lexicographic order on $M \times \mathbb{P}(\mathbb{C}^n)$ associated to a partial order $\preceq_M$ on M and local isocones $I_x \in \mathcal{I}_n$. Then there is an isocone I in A inducing $\preceq$ iff

- $\prec_M$ is closed.
- $L : x \mapsto I_x$ is lower hemi-continuous.





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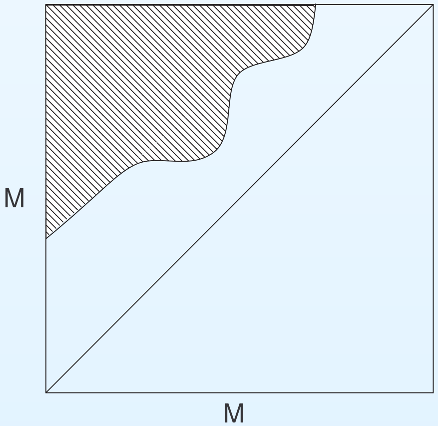
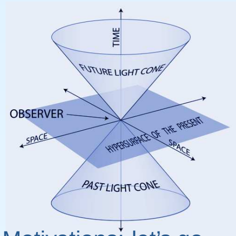


$\longrightarrow$ Causality disappears at small scale !



Perspectives and other wild conjectures: let's go crazy !

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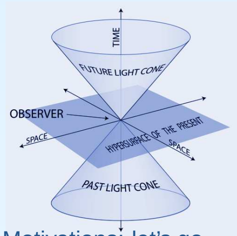

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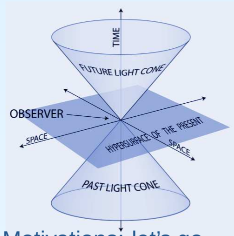

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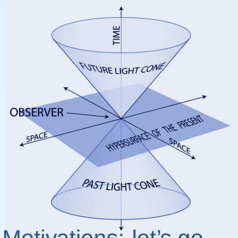

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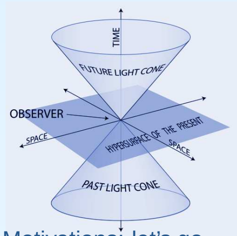
- Classification of isocones related with the strong CP problem ?



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- N. Bizi, F. Besnard, work in progress