Categorical Operator Algebraic Foundations of Relational Quantum Theory

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Abstract 1

We provide an algebraic formulation of C.Rovelli's relational quantum theory[1] that is based on suitable notions of "non-commutative" higher operator categories, originally developed in the study of categorical non-commutative geometry.[2] [3] [4]

[1]Rovelli C (1996)
Relational Quantum Mechanics
Int J Theor Phys 35:1637 [arXiv:quant-ph/9609002].

[2]B P, Conti R, Lewkeeratiyutkul W (2007)
Non-commutative Geometry, Categories and Quantum Physics
East-West Journal of Mathematics 2007:213-259 [arXiv:0801.2826v2].

[3]B P, Conti R, Lewkeeratiyutkul W (2012)
Categorical Non-commutative Geometry
J Phys: Conf Ser 346:012003.

[4]B P, Conti R, Lewkeeratiyutkul W, Suthichitranont
Strict Higher C*-categories preprint(s) (to appear).

Abstract 2

As a way to implement C.Rovelli's original intuition on the relational origin of space-time,[5] in the context of our proposed algebraic approach to quantum gravity via Tomita-Takesaki modular theory,[6] we tentatively suggest to use this categorical formalism in order to spectrally reconstruct non-commutative relational space-time geometries from categories of correlation bimodules between operator algebras of observables.

[5] Rovelli C (1997)
Half Way Through the Woods
The Cosmos of Science 180-223
Earman J, Norton J (eds) University of Pittsburgh Press.

[6] B P, Conti R, Lewkeeratiyutkul W (2010)
Modular Theory, Non-commutative Geometry and Quantum Gravity
SIGMA 6:067 [arXiv:1007.4094v2].

Abstract 3

Part of this work is a joint collaboration with:

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- Assoc.Prof.Wicharn Lewkeeratiyutkul (Chulalongkorn University),
- Dr.Matti Raasakka (Paris 13 University)
- Dr.Noppakhun Suthichitranont.

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  - Examples: Relations, Hypermatrices, Hyper C*-algebras

$\star\star$ Relational Spectral Space-Time

Ideology

- *covariance and dynamic laws are both described by (higher) categorical structures of "correlations" between "observers";*
- *higher-C*-categories are a possible algebraic quantum mathematical formalism for the study of C.Rovelli's "relational quantum mechanics";*
- *a "physical system" is completely captured by the (higher) categorical structure of "correlations";*
- *the physical geometry (space-time) of the system is determined by the base category of the (higher) categorical bundle of these "interaction/correlations" between observable algebras;*
- *...such a non-commutative space-time organization of the system is spectrally recovered via "relational" modular theory.*

- **Relational Quantum Theory - Background**

Relationalism: Dynamics = Correlations

- Relationalism in physics has a long tradition: G.Leibniz, G.Berkeley, E.Mach, ...
- Relational dynamics is a core feature of Einstein's theory of relativity (special and general): the dynamics is not specified as an explicit functional evolution with respect to a time parameter, but it is given by an implicit relation between the several variables (Rovelli's partial observables).
- Similarly (Einstein's hole argument), localization of events in general relativity is not absolute: coordinates are gauge and points on a Lorentz manifold are not objective elements of the theory (coincidences, events and correlations, that are preserved by local diffeomorphisms, are).

Functions / Relations

In this classical context, mathematically speaking, the transition is between functions and relations (more generally 1-quivers):

function	relation or quiver
$t \mapsto F(t)$	$s(\tau) \longleftarrow\!\dashv \tau \vdash\!\longrightarrow t(\tau)$
$t \xrightarrow{\quad t \quad} F(t)$	$\begin{tikzcd} s(\tau) \arrow[r, bend left, "\tau'"] \arrow[r, bend right, "\tau"'] & t(\tau) \end{tikzcd}$
$F : A \to B$	$R : T \to A \times B$

Relational Quantum Theory 1

In 1994, C.Rovelli elaborated **relational quantum mechanics** as an attempt to radically solve the interpretational problems of quantum theory.[7] This approach is based on two assumptions:

- **relativism**: all systems (necessarily quantum) have equivalent status, there is no difference between observers and objects.
- **completeness**: quantum physics is a complete and self-consistent theory of natural phenomena.

[7] Rovelli C (1996) **Relational Quantum Mechanics** *Int J Theor Phys* 35:1637 [arXiv:quant-ph/9609002].

Relational Quantum Theory 2

Analysis of the Schrödinger's cat problem entails:

- states are **relative** to each observer: different observers can give different (but "compatible") accounts of the interactions.
- the only physical properties (interactions) are **correlations** between observers.
- physics is about **information** exchange between agents: correlations describe the "relative information" that observers posses about each other.

Quantum and Relativistic Relationalisms

In 1996 C.Rovelli went further with the radical conjecture:[8]

- there is a direct connection between:
  - quantum relationalism via correlations of systems,
  - general relativistic relational status of space-time localization determined by contiguity of events.
- This strongly suggests that it should be possible to reinterpret the information on space-time localization (contiguity) as correlations (interactions) between quantum systems, opening the way for a reconstruction of space-time "a-posteriori" from purely quantum correlations (see also R.Haag 1990).

It is our purpose to provide some mathematical implementation in support of this approach to quantum relativity.

[8] Rovelli C (1997) **Half Way Through the Woods** *The Cosmos of Science* 180-223 Earman J, Norton J (eds) University of Pittsburgh Press.

Higher C*-categories in Relational Quantum Theory

- A general mathematical framework for relational physics is still missing and we propose to formalize "correlations" (relations between quantum systems) and their "compatibility" using a higher C*-categorical environment:
  - Systems and observers are represented by C*-algebraic data.
  - Correlations and interactions are represented by "bimodules".
  - There is a modular hierarchy of systems in mutual correlation, because we must distinguish "observers" from "observers of observers", 'observers of observers of observers" and so on ...
  - The mutual compatibility requested is encoded by the covariance coming from the compositions operations of an higher category.
  - Systems with higher internal correlations are described by hyper-C*-algebras.

• Quantum Correlations

- Quantum systems as C*-algebras,
- Spectra of quantum spaces and their morphisms,
- Correlations as suitable bivariant bimodules,
- Higher correlations ...

Quantum Phase Spaces = C*-algebras of Observables

Following the usual framework of **algebraic quantum theory**,[9] we accept as tentative assumption that:

- quantum systems can be described as **C*-algebras**.
- classical systems, as a special case, are described by commutative C*-algebras.

Gel'fand-Naĭmark duality assures that every commutative C*-algebra $\mathcal{A}$ is $*$-isomomorphic to the algebra $C(\mathrm{Sp}(\mathcal{A}))$ of continuous functions over its spectrum $\mathrm{Sp}(\mathcal{A})$ that is a locally compact Hausdorff topological space (phase space):

$$\begin{aligned} \textit{Classical space} &= \textit{spectrum of C*-algebra} \\ &\simeq \textit{locally compact Hausdorff space} \end{aligned}$$

[9] Strocchi F (2005) **An Introduction to the Mathematical Structure of Quantum Mechanics: a Short Course for Mathematicians** World Scientific.

Quantum Spectra (conjectural)

Motivated by our work on spectral theory of commutative full C*-categories[10] we find inspiration in the following:

Spectral Conjecture:
there is a spectral theory of non-commutative C-algebras in term of families of Fell complex line-bundles over involutive categories.*

Quantum space = *spectrum of C*-algebra*
= *Fell line-bundle over an involutive category*

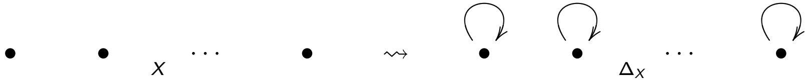
Quantum correlations = *morphisms of quantum spaces*
= *spectra of (higher) bimodules*
= *(higher) quivers.*

[10] B P, Conti R, Lewkeeratiyutkul W (2011) **A Horizontal Categorification of Gel'fand Duality** *Adv Math* 226(1):584-607 [arXiv:0812.3601v2].

Morphisms of Non-commutative Spaces 1

$$\begin{aligned} \text{Classical space } X &= \text{spectrum of Abelian C*-algebra } C(X;\mathbb{C}) \\ &= \text{trivial line bundle } X\times\mathbb{C} \text{ over space } X \\ &= \text{Fell line-bundle over the space } \Delta_X \text{ of ``loops'' of } X \end{aligned}$$

$$\begin{aligned} \text{Abelian C*-algebra } C(X) &= \text{algebra } \Gamma(X; X\times\mathbb{C}) \text{ of sections of } X\times\mathbb{C} \\ &= \text{convolution algebra } \Gamma(\Delta_X; \Delta_X\times\mathbb{C}) \end{aligned}$$



$\Delta_X \times \mathbb{C} =$

Morphisms of Non-commutative Spaces 2

$$\begin{aligned}\text{Morphism of classical spaces } X, Y &= \text{map/relation/1-quiver} : X \to Y \\ &= \text{level-2 relation} : \Delta_X \to \Delta_Y\end{aligned}$$

$$x \longmapsto y \qquad \rightsquigarrow \qquad x \circlearrowright \;\Rightarrow\; \circlearrowleft y \qquad x \in X, \ y \in Y.$$

For a relation $R \subset X \times Y$ (1-quiver) with reciprocal $R^* \subset Y \times X$, the "convolution algebra" $\mathcal{A}$ of the Fell line-bundle with base $\Delta_X \cup R \cup R^* \cup \Delta_Y$ contains the C*-algebras $C(X)$, $C(Y)$, a bimodule $\Gamma(R, R \times \mathbb{C})$ and its contragredient $\Gamma(R^*; R^* \times \mathbb{C})$.

$$\mathcal{A} = \begin{bmatrix} C(X) & \Gamma(R^* \times \mathbb{C}) \\ \Gamma(R \times \mathbb{C}) & C(Y) \end{bmatrix}$$

Hence the morphisms from X to Y are dually given by (Hilbert C*) bimodules over $C(Y)$ and $C(X)$.

Morphisms of Non-commutative Spaces 3

Quantum space = space of points with "relations" = 1-quiver Q^1

Algebra of functions on Q^1 = *"convolution" algebra of* Q^1

As a consequence we claim that:

- at the "spectral level" a morphism between two quantum spaces Q^1_X and Q^1_Y is a 2-quiver Q^2 with 2-cells like

$$\begin{tikzcd} x_1 \arrow[r, dotted] \arrow[d, bend right, "f"'] \arrow[dr, dotted] & y_1 \arrow[d, bend left, "g"] \\ x_2 \arrow[r, dotted] & y_2 \end{tikzcd} \qquad f \in Q^1_X, \quad g \in Q^1_Y,$$

- at the "dual level" a morphism of quantum spaces is a "level-2 bimodule" inside the convolution depth-2 hyper C*-algebra $\Gamma(Q^2)$ of the involutive 2-category generated by the morphism 2-quiver Q^2.

Obstacle: we need a "bivariant" notion of Hilbert C*-bimodule !

Quantum Correlations as Bimodules - Relational Networks

Correlations = "bimodules":

- Inclusions of subsystems (homomorphisms) and symmetries (isomorphisms) $\phi : \mathcal{A} \to \mathcal{B}$ give adjoint pairs of twisted bimodules ${}_{\phi}\mathcal{B}, \mathcal{B}_{\phi}$.
- States ω on $\mathcal{A}$, via GNS-representation $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$, give bimodules ${}_{\mathcal{A}}(\mathcal{H}_\omega)_{\mathbb{C}}$.
- Conditional expectations $\Phi : \mathcal{A} \to \mathcal{B}$ give $\mathcal{A}$-$\mathcal{B}$ bimodules via Kasparov GNS-representation theorem.

Rovelli's "relational network":
"Physical system = 1-categorical structure of correlations"

Correlations from Symmetries

Two observer systems (C*-algebras) can be related by symmetries.

The usual case is given by the “geometrical symmetries” of the algebras of localized observables in algebraic quantum field theory.

If $\mathcal{O}_1, \mathcal{O}_2$ are two regions in Minkowski space-time and g is an element of the Poincaré group such that $g(\mathcal{O}_1) = \mathcal{O}_2$, the axiom of Poincaré covariance assure the existence of an isomorphism $\alpha_g : \mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2)$ between the C*-algebras localized in $\mathcal{O}_1$, $\mathcal{O}_2$.

Dynamics, as long as it is implemented via unitary evolution, can be seen as a special case of these geometrical correlations (time translations) connecting observers at different times $\mathcal{O} \mapsto \mathcal{O} + t$.

Symmetries as Twisted Bimodules

In algebraic quantum theory (following Wigner), symmetries are described by linear isomorphisms (or conjugate-linear anti-isomorphisms) $\phi : \mathcal{A} \to \mathcal{B}$ between two C*-algebras of observables.

To every such symmetry ϕ, there is a naturally associated adjoint pair of $\mathcal{A}$-$\mathcal{B}$ bimodules ${}_\phi\mathcal{B}$ and $\mathcal{B}_\phi$ obtained by left or right ϕ-twisting of the product in $\mathcal{B}$:

$$a \cdot x \cdot b := \phi(a)xb, \quad \forall a \in \mathcal{A},\ b \in \mathcal{B},\ x \in {}_\phi\mathcal{B},$$
$$b \cdot x \cdot a := bx\phi(a), \quad \forall a \in \mathcal{A},\ b \in \mathcal{B},\ x \in \mathcal{B}_\phi,$$

Composition of symmetries functorially corresponds to the internal tensor product of bimodules:

$$\mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \quad \mapsto \quad \mathcal{C}_{\psi\circ\phi} \simeq \mathcal{C}_\psi \otimes_{\mathcal{B}} \mathcal{B}_\phi.$$

Correlations from Localization

Any unital inclusion of C*-algebras or more generally any unital $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between unital C*-algebras, will provide a correlation via the ϕ-twisted bimodule ${}_{\phi}\mathcal{B}$.

In this way we see that "localization" can be formalized on the same footing as covariance using correlation bimodules.

Strictly speaking we do not obtain a C*-category, because we can have more than one correlation between the same systems, such generalization of a C*-category is called a Fell bundle.

Correlations from States and Conditional Expectations

In quantum physics we have a further type of correlation between observers that is responsible for the second form of dynamical evolution, via "collapse of the wave function" as well as for the "quantum channels" of quantum information theory.

These "interactive correlations" as usually formalized in quantum mechanics via completely positive maps between observable algebras, states and conditional expectations are special cases.

Correlations via states and conditional expectations can also be reformulated using bimodules between algebras of observers.

States and Conditional Expectations as GNS-Bimodules

In algebraic quantum theory, a state is given by a normalized positive complex-linear functional $\omega : \mathcal{A} \to \mathbb{C}$ on a (unital) C*-algebra of observables $\mathcal{A}$.

To every such state, we can naturally associate a $\mathcal{A}$-$\mathbb{C}$ bimodule $\mathcal{H}_\omega$ via the usual Gel'fand-Naĭmark-Segal representation theorem:

- $\mathcal{H}_\omega$ is the Hilbert space (i.e. $\mathbb{C}$-bimodule) obtained by separation and completion of the vector space $\mathcal{A}$ under the inner product $\langle x \mid y \rangle := \omega(x^* y)$, with $x, y \in \mathcal{A}$,
- $\mathcal{H}_\omega$ is a left $\mathcal{A}$-module via the representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ obtained by linear continuous extension (of the quotient by the null space) of the left action of $\mathcal{A}$ on itself.

Conditional expectations are similary associated to a bimodule (C*-correspondence) via G.Kasparov GNS representation theorem.

Physical Systems = Categories of Correlations 1

Different observers are now mutually related by a family of quantum correlation channels, some of them describing symmetries, others quantum interactions.

Each observer is still equipped with a family of potential states, but now states of different observers can be compared via the family of binary correlations so far introduced.

The dynamic of the quantum theory has been totally codified via correlations and the potentially huge Cartesian product of state-spaces of the observers is now collapsed to a much more manageable set of states that are compatible under the given correlations.

Physical Systems = Categories of Correlations 2

A a first step in the mathematical formalization of C.Rovelli relational quantum mechanics we propose the following statement:

- A physical system is totally captured by such a "category" of bimodules of binary correlations
(C.Rovelli's "relational network").

In principle we might also try to consider *n*-tuple correlations between observers. Multimodules and their C*-polycategories would be necessary to formalize mathematically such notions.[11]

A physical system is for now formalized as a 1-categorical structure: level-1 correlations between algebras of observables of different agents. Do we need higher categories?

[11] C*-polycategories, work in progress.

Its Observers All The Way Up ...

The "vertical categorification catastrophy" is almost inescapable:

- mathematically, the family of physical systems itself is a 2-category (via functors and natural transformations);
- the ideological assumption of role interchangeability between systems and observers requires that such higher categories should be physically relevant: the systems must themseves be observers, object of further correlations;
- correlations between two systems could in principle be reconduced to lower level correlations between their "internal agents", but this **reductionist approach** is not compatible with the original introduction of observers as "black building blocks" whose internal correlation structure is "not affected" by the (several alternative) external quantum correlations!

Higher Correlations

- Given two quantum systems $\mathcal{A}, \mathcal{B}$, a pair of observers can give a different description of their interaction correlations $\mathcal{M}, \mathcal{N}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{M}"] & \mathcal{B} \arrow[l, bend left, "\overline{\mathcal{M}}"] \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{N}"] & \mathcal{B} \arrow[l, bend left, "\overline{\mathcal{N}}"] \end{tikzcd}.$$

- The mutual compatibility between the two correlations is decribed by a "higer level" morphism $\Phi : \mathcal{M} \to \mathcal{N}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathcal{M}"{name=U}] \arrow[r, bend right, "\mathcal{N}"'{name=D}] \arrow[r, phantom, "\Downarrow\Phi"] & \mathcal{B} \end{tikzcd}.$$

- The morphism Φ can be seen as a (level-2) correlation bimodule between the C*-algebroids $\mathcal{T}(\mathcal{M}), \mathcal{T}(\mathcal{N})$ generated respectively by $\mathcal{M}$ and $\mathcal{N}$. And so on ...

Observers of Observers of ... aka "The Hyper-Matrix" :-)

This opens the way to the scary possibility to have different levels of "reality" for quantum properties, since observers and systems are now not only "extensively" related, but "hierarchically" structured.

Higher categories will be necessary to formalize this situation.

- One might propose a **hypercovariance principle** to deal with the invariance of the physics along the hierarchical ladder of observers/systems.
- Higher C*-categories and higher Fell bundles have been developed from the beginning with this kind of goals in mind and can potentially deal with such a context of interacting "structured virtual realities".

• "Quantum" Higher Categories

- Mathematical obstacle 1:
  usual higher category theory cannot accommodate in a non-trivial way non-commutativity (quantum subsystems)!
- Mathematical obstacle 2:
  to describe higher level relational situations we need to develop a theory of involutions for higher categories.

Globular Higher Arrows and Their Compositions

- 0-arrows (objects): $\bullet$, 1-arrows: $\bullet \longrightarrow \bullet$

  $\circ^1_0$-composition: $A \xrightarrow{g} B \xrightarrow{f} C \rightsquigarrow A \xrightarrow{f \circ^1_0 g} C$

- 2-arrows:
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

  $\circ^2_0$-composition:
$$\begin{tikzcd} A \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\Psi"] & B \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\Phi"] & C \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[rr, bend left, "f_1 \circ^1_0 g_1"] \arrow[rr, bend right, "f_2 \circ^1_0 g_2"'] \arrow[rr, phantom, "\Downarrow\Phi \circ^2_1 \Psi"] & & C \end{tikzcd}$$

  $\circ^2_1$-composition:
$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, bend left=30, "\Downarrow\Theta"] \arrow[r, phantom, bend right=30, "\Downarrow\Lambda"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\Downarrow\Lambda \circ^2_1 \Theta"] & & B \end{tikzcd}$$

Cubical Higher Arrows and Their Compositions

- 2-arrows:
$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, dotted] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

- $\circ^2_h$-composition:
$$\begin{tikzcd} A_{11} \arrow[r, "f_{11}"] \arrow[d, "g_1"'] \arrow[dr, dotted, "\Phi"] & A_{12} \arrow[r, "f_{12}"] \arrow[d, "g_2"] \arrow[dr, dotted, "\Psi"] & A_{13} \arrow[d, "g_3"] \\ A_{21} \arrow[r, "f_{21}"'] & A_{22} \arrow[r, "f_{22}"'] & A_{23} \end{tikzcd} \mapsto \begin{tikzcd} A_{11} \arrow[r, "f_{12}\circ^1_h f_{11}"] \arrow[d, "g_1"'] \arrow[dr, dotted, "\Psi\circ^2_h\Phi"] & A_{13} \arrow[d, "g_3"] \\ A_{21} \arrow[r, "f_{22}\circ^1_h f_{21}"'] & A_{23} \end{tikzcd}$$

- $\circ^2_v$-composition
$$\begin{tikzcd} A_{11} \arrow[r, "f_1"] \arrow[d, "g_{11}"'] \arrow[dr, dotted, "\Phi"] & A_{12} \arrow[d, "g_{12}"] \\ A_{21} \arrow[r, "f_2"] \arrow[d, "g_{21}"'] \arrow[dr, dotted, "\Psi"] & A_{22} \arrow[d, "g_{22}"] \\ A_{31} \arrow[r, "f_3"] & A_{32} \end{tikzcd} \mapsto \begin{tikzcd} A_{11} \arrow[r, "f_1"] \arrow[d, "g_{21}\circ^1_v g_{11}"'] \arrow[dr, dotted, "\Psi\circ^2_v\Phi"] & A_{12} \arrow[d, "g_{22}\circ^1_v g_{12}"] \\ A_{31} \arrow[r, "f_3"'] & A_{32} \end{tikzcd}$$

Globular Strict Higher Categories

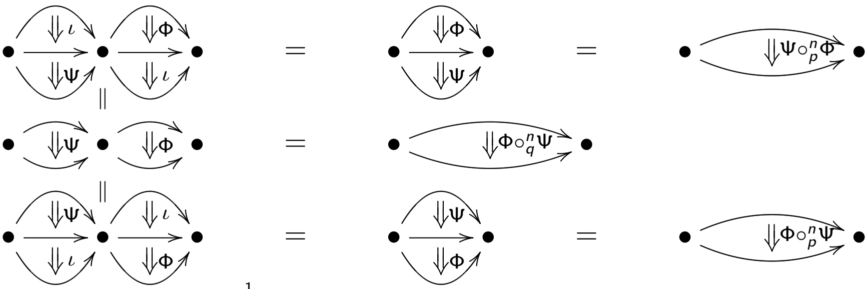
A **globular** n**-category** $(\mathcal{C}, \circ_0, \cdots \circ_{n-1})$ is a family $\mathcal{C}$ of n-arrows equipped with a family of partially defined binary compositions $\circ_p$, for $p := 0, \dots, n-1$, that satisfy the following list of axioms:

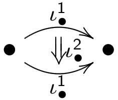
- for all $p = 0, \dots, n-1$, $(\mathcal{C}, \circ_p)$ is a 1-category, whose partial identities are denoted by $\mathcal{C}^p$,
- for all $q < p$, a $\circ_q$-identity is also a $\circ_p$-identity: $\mathcal{C}^q \subset \mathcal{C}^p$,
- for all $p, q = 0, \dots n-1$, with $q < p$, the $\circ_q$-composition of $\circ_p$-identities, whenever exists, is a $\circ_p$-identity: $\mathcal{C}^p \circ_q \mathcal{C}^p \subset \mathcal{C}^p$,
- the **exchange property** holds for all $q < p$: whenever $(x \circ_p y) \circ_q (w \circ_p z)$ exists also $(x \circ_q w) \circ_p (y \circ_p z)$ exists and they coincide.[12]

[12] By symmetry, the exchange property automatically holds for all $q \neq p$.

Eckmann-Hilton Collapse

For $q < p < n$ and n-arrows with a common q-source q-target $\bullet$:
$\circ^n_p = \circ^n_q$ and they are commutative operations!



where ι is 

Non-commutative Exchange: Globular Case

$$
\begin{tikzcd}
A \arrow[r, bend left, "e"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\iota_e"] & B \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\Psi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Phi"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, bend left=70, "e"] \arrow[r, "e"] \arrow[r, bend right=70, "e"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\iota_e"] \arrow[r, phantom, shift right=3ex, "\Downarrow\iota_e"] & B \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\Psi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Phi"] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, bend left=50, "f\circ e"] \arrow[rr, "g\circ e"] \arrow[rr, bend right=50, "h\circ e"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\Psi\circ\iota_e"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\Phi\circ\iota_e"] & & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\Psi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Phi"] & C \arrow[r, bend left, "e"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\iota_e"] & D
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, bend left=70, "f"] \arrow[r, "g"] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\Psi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Phi"] & C \arrow[r, bend left=70, "e"] \arrow[r, "e"] \arrow[r, bend right=70, "e"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\iota_e"] \arrow[r, phantom, shift right=3ex, "\Downarrow\iota_e"] & D
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[rr, bend left=50, "e\circ f"] \arrow[rr, "e\circ g"] \arrow[rr, bend right=50, "e\circ h"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\iota_e\circ\Psi"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\iota_e\circ\Phi"] & & D
\end{tikzcd}
$$

non-commutative exchange:
for all p-identities ι, for all $q < p$, the partially defined maps $\iota \circ_q - : (\mathcal{C}, \circ_p) \to (\mathcal{C}, \circ_p)$ and $- \circ_q \iota : (\mathcal{C}, \circ_p) \to (\mathcal{C}, \circ_p)$ are functorial (homomorphisms of partial 1-monoids).

Duals of Globular n-arrows

- Duals of 1-arrows: $A \xrightarrow{f} B \quad \mapsto \quad A \xleftarrow{f^*} B$
- Duals of 2-arrows:

$$*_1 : \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\Phi"] & B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Uparrow\Phi^{*_1}"] & & B \end{tikzcd}$$

$$*_0 : \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\Phi"] & B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A & & B \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "g^*"] \arrow[ll, phantom, "\Downarrow\Phi^{*_0}"] \end{tikzcd}$$

$$*_{0,1} : \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\Phi"] & B \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A & & B \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "g^*"] \arrow[ll, phantom, "\Uparrow\Phi^{*_{0,1}}"] \end{tikzcd}$$

- For n-arrows we have 2^n duals $*_\alpha$ (including the identity) exchanging q-sources / q-targets for q in an arbitrary set $\alpha \subset \{0, \dots, n-1\}$.

Duals of Cubical n-arrows

- $*^2_h$:

$$\begin{tikzcd} A_{11} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, dotted, "\Phi"] & A_{12} \arrow[d, "g_2"] \\ A_{21} \arrow[r, "f_2"'] & A_{22} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_{11} \arrow[d, "g_1"'] & A_{12} \arrow[l, "f_1^{*^1_h}"'] \arrow[d, "g_2"] \arrow[dl, dotted, "\Phi^{*^2_h}"] \\ A_{21} & A_{22} \arrow[l, "f_2^{*^1_h}"] \end{tikzcd}$$

- $*^2_v$:

$$\begin{tikzcd} A_{11} \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, dotted, "\Phi"] & A_{12} \arrow[d, "g_2"] \\ A_{21} \arrow[r, "f_2"'] & A_{22} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A_{11} \arrow[r, "f_1"] & A_{12} \\ A_{21} \arrow[u, "g_1^{*^1_v}"] \arrow[r, "f_2"'] \arrow[ur, dotted, "\Phi^{*^2_v}"] & A_{22} \arrow[u, "g_2^{*^1_v}"'] \end{tikzcd}$$

- For a cubical n-category we have (including the identity) 2^n possible duals $*_\alpha$ (preserving the composability class of the n-cells), one for every subset $\alpha \subset \{d_1, \dots, d_n\}$ of directions of the edges of the n-cubical cells.

Higher Involutions

We have an **involutive (higher) category** whenever:
we have some duality maps $*_\alpha$, with $\alpha \subset \{0, \dots, n-1\}$ that are:

- covariant functors for all $\circ_q$, with $q \notin \alpha$,
- contravariant functors for all $\circ_q$, with $q \in \alpha$,
- involutive: $(x^{*_\alpha})^{*_\alpha} = x$,
- Hermitian:[13] $x^{*_\alpha} = x$, for all $\circ_q$-identities, with $q = \min(\alpha)$,
- commuting: $(x^{*_\alpha})^{*_\beta} = (x^{*_\beta})^{*_\alpha}$.

The higher category $(\mathcal{C}, \circ_0, \dots, \circ_{n-1})$ is **fully involutive** if its family of involutions generates all possible 2^n dualities of n-arrows.

[13] This is the statement for the globular case.

Quantum Globular Strict Higher C*-Categories

A **"quantum" fully involutive strict globular** n-**C*-category** $(\mathcal{C}, \circ_0, \dots, \circ_{n-1}, *_0, \dots, *_{n-1}, +, \cdot, \|\cdot\|)$ is a fully involutive strict n-category with **non-commutative exchange** such that:

- for all $a, b \in \mathcal{C}^{n-1}$, the fiber $\mathcal{C}_{ab}$ is Banach with norm $\|\cdot\|$,[14]
- for all p, $\circ_p$ is fiberwise bilinear and $*_p$ is conjugate-linear,
- for all $\circ_p$, $\|x \circ_p y\| \leq \|x\| \cdot \|y\|$, whenever $x \circ_p y$ exists,
- for all p, $\|x^{*_p} \circ_p x\| = \|x\|^2$, for all $x \in \mathcal{C}$,
- for all p, $x^{*_p} \circ_p x$ is positive in the C*-algebra envelope of $\mathcal{C}_{ee}$ $(\mathcal{E}(\mathcal{C}_{ee}), \circ_p, *_p, +, \cdot, \|\cdot\|)$, where e is the p-source of x.

A **partially involutive strict** n-**C*-category** will be equipped with only a subfamily of the previous involutions and will satisfy only those properties that can be formalized using the given involutions.

[14] By definition $\mathcal{C}_{ab} := \{x \in \mathcal{C} \mid b \circ_{n-1} x, \quad x \circ_{n-1} a \quad \text{both exist}\}$.

Matrices = Convolution Algebras of Pair Groupoids

"A square matrix $[x^i_j] \in \mathbb{M}_{N\times N}(\mathbb{C})$ is a section of a Fell line-bundle $\mathcal{E}$ over a discrete finite pair groupoid $\mathcal{X} : \mathcal{X}^1 \rightrightarrows \mathcal{X}^0$":

$$\begin{tikzcd} 1 \arrow[loop left, "(1,1)"] \arrow[r, bend left, "(N,1)"] & N \arrow[l, bend left, "(1,N)"] \arrow[loop right, "(N,N)"] \end{tikzcd}$$

- a finite set of **objects** $\mathcal{X}^0 := \{1, \dots, N\}$
- a finite set of **1-arrows** (ordered pairs) $(i,j) \in \mathcal{X}^1 := \mathcal{X}^0 \times \mathcal{X}^0$, with source j and target i
- for every 1-arrow $(i,j) \in \mathcal{X}^1$, a **fiber** $\mathcal{E}_{ij}$ over (i,j) that is just a copy of the set $\mathbb{C}$ of complex numbers
- a **section** i.e. a function $x : \mathcal{X}^1 \to \mathcal{E} := \bigcup_{(i,j)\in\mathcal{X}^1} \mathcal{E}_{ij}$ such that $x^i_j := x(i,j) \in \mathcal{E}_{ij}$, for all $(i,j) \in \mathcal{X}^1$
- the same construction works with an associative involutive algebra $\mathcal{A}$ in place of $\mathbb{C}$: $\mathbb{M}_{N\times N}(\mathcal{A}) \simeq \mathbb{M}_{N\times N}(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}$.

Hyper Convolutions Algebras

- The same construction can be generalized taking any finite globular n-category[15] $(\mathcal{X}, \circ_0, \dots, \circ_{n-1})$ in place of $N \times N$ and any associative unital complex $*$-algebra $\mathcal{A}$ in place of $\mathbb{C}$.
- The family of sections $\mathbb{M}_{\mathcal{X}}(\mathcal{A})$ of the bundle $\mathcal{E} := \mathcal{X} \times \mathcal{A}$ is a convolution algebra with n operations and n involutions:
  $(\sigma \circ_p \rho)_z := \sum_{x \circ_p y = z} \sigma_x \cdot_{\mathcal{A}} \rho_y$,
  $(\sigma^{*_p})_z := (\sigma_{z^{*_p}})^{*_{\mathcal{A}}}$, for all $p \in A \subset \{0, \dots, n-1\}$.
- $\mathcal{E} \subset \mathbb{M}_{\mathcal{X}}(\mathcal{A})$ is a strict globular involutive n-category.

We can think of the sections $\sigma \in \mathbb{M}_{\mathcal{X}}(\mathcal{A})$ as "hypermatrices" whose entries $\sigma_x \in \mathcal{A}$ are indexed by n-arrows in a globular strict finite A-involutive n-category $\mathcal{X}$ in place of the pair groupoid $\{1, \dots, N\} \times \{1, \dots, N\}$.

[15] With usual exchange law or with non-commutative exchange.

Hyper C*-algebras

Definition

A **hyper C*-algebra** $(\mathcal{A}, \circ_0, \dots, \circ_{n-1}, *_0, \dots, *_{n-1})$ will be a complete topo-linear space $\mathcal{A}$ equipped with different pairs of multiplication/involution $(\circ_k, *_k)$, for $k = 0, \dots n-1$, inducing on $\mathcal{A}$ a C*-algebra structure, via a necessarily unique C*-norm $\| \cdot \|_k$.

Proposition

Given unital C-algebra $\mathcal{A}$ and a finite globular (cubical) higher (fully) involutive n-category $\mathcal{X}$, the $\mathcal{X}$-convolution $*$-algebra $\mathbb{M}_{\mathcal{X}}(\mathcal{A})$ is a hyper C*-algebra with the operations of $\circ_q$-convolution and $*_q$-involutions, for $q = 0, \dots, n-1$.*

Hypermatrices 1

Elementary finite dimensional examples come from hypermatrices: $\mathbb{M}_{\mathcal{X}_1\times\cdots\times\mathcal{X}_n}(\mathbb{C}) := \mathbb{M}_{\mathcal{X}_1}(\cdots\mathbb{M}_{\mathcal{X}_n}(\mathbb{C})\cdots) \simeq \mathbb{M}_{\mathcal{X}_1}(\mathbb{C})\otimes\cdots\otimes\mathbb{M}_{\mathcal{X}_n}(\mathbb{C})$, where $\mathcal{X}_1,\dots\mathcal{X}_n$ are here finite involutive 1-categories and in particular when $\mathcal{X}_k := \{1,\dots,N_k\}\times\{1,\dots,N_k\}$, $k=1,\dots,n$ are finite pair groupoids.

A **hypermatrix of depth-n** is a multimatrix $[x^{j_1\cdots j_n}_{i_1\dots i_n}] \in \mathbb{M}_{N_1^2\dots N_n^2}(\mathbb{C})$ with indexes $i_k, j_k = 1,\dots N_k$, for all $k=1,\dots,n$.

- on $\mathbb{M}_{N_1^2\dots N_n^2}(\mathbb{C})$ there are 2^n different multiplications: acting at every level either as convolution or as Schur product: $[x^{i_1\dots i_k\dots i_n}_{j_1\dots j_k\dots j_n}] \bullet_\gamma [y^{i'_1\dots i'_k\dots i'_n}_{j'_1\dots j'_k\dots j'_n}] := [\sum_{k\in\gamma}\sum_{o_k=1}^{N_k} x^{i_1\dots i_k\dots i_n}_{j_1\dots o_k\dots j_n}\ y^{i_1\dots o_k\dots i_n}_{j_1\dots j_k\dots j_n}]$ where $\gamma\subset\{1,\dots,n\}$ is the set of contracting indexes.

Hypermatrices 2

- there are 2^n involutions taking the conjugate of all the entries and, at every level, either the transposed or the identity: $[x^{i_1\dots i_k\dots i_n}_{j_1\dots j_k\dots j_n}]^{\star_\gamma} := [\overline{x}^{i_1\dots j_{k_1}\dots j_{k_m}\dots i_n}_{j_1\dots i_{k_1}\dots i_{k_m}\dots j_n}]$, for all $\gamma := \{k_1, \dots, k_m\} \subset \{1, \dots, n\}$.
- there are 2^n C*-norms taking either the operator norm or the maximum norm at every level: using the natural isomorphism $\mathbb{M}_{N_1^2\dots N_n^2}(\mathbb{C}) \simeq \mathbb{M}_{N_1^2}(\mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{M}_{N_n^2}(\mathbb{C})$, $\forall \gamma \subset \{1, \dots, n\}$, $\|[x^{i_1}_{j_1}] \otimes \cdots \otimes [x^{i_n}_{j_n}]\|_\gamma := \prod_{k\in\gamma} \|[x^{i_k}_{j_k}]\| \cdot \prod_{k'\notin\gamma} \|[x^{i_{k'}}_{j_{k'}}]\|_\infty$, where $\|[x^{i_k}_{j_k}]\|$ is the C*-norm on $\mathbb{M}_{N_k}(\mathbb{C})$ and $\|[x^{i_k}_{j_k}]\|_\infty := \max_{i,j} |x^i_j|$.

$(\mathbb{M}_{N_1^2\dots N_n^2}(\mathbb{C}), \bullet_\gamma, \star_\gamma, \| \ \|_\gamma, \gamma \subset \{1, \dots, n\})$ is a hyper C*-algebra.

Hypermatrices can be seen as convolution hyper C*-algebras of product cubical n-categories with "extra" compositions.

• Relational Spectral Space-Time

The formalization of relational quantum theory via higher C*-categories is only the second intermediate step in our ongoing research program on **modular algebraic quantum theory**

Modular Algebraic Quantum Theory 1 (Ideology)

* **quantum theory** *is a fundamental theory of physics and should not come from a quantization;*
* **geometry** *should be* **spectrally reconstructed a posteriori** *from a basic operational theory of observables and states;*
* **A.Connes' non-commutative geometry** *provides the natural environment where to attempt an implementation of the spectral reconstruction of a "quantum" space-time;*
* **Tomita-Takesaki modular theory** *should be the main tool to achieve the previous goals, associating to operational data, spectral non-commutative geometries;*
* **categories of operational data** *provide the general framework for the formulation of covariance in this context ... and ultimately for the identification of the geometric degrees of freedom (space-time) hidden in the theory.*

Modular Algebraic Quantum Theory 2

In our **modular algebraic quantum gravity** program:[16]

- Every state ω on a C*-algebra $\mathcal{O}$ of partial observables induces a net of subalgebras $\mathcal{A} \subset \mathcal{O}$ such that $\omega|_{\mathcal{A}}$ is a KMS-state.
- By **Tomita-Takesaki theory**, every such KMS-state ω on a C*-subalgebra $\mathcal{A}$ uniquely determines a modular spectral **non-commutative geometry** $(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega)$ where:
  - $\mathcal{H}_\omega$ is the Hilbert space of the GNS representation π_ω induced by $\omega|_{\mathcal{A}}$, with cyclic separating unit vector $\xi_\omega \in \mathcal{H}_\omega$,
  - $K_\omega := \log \Delta_\omega$ is the generator of the one-parameter unitary group $t \mapsto \Delta_\omega^{it}$ spatially implementing the modular one-parameter group of $*$-automorphisms $\sigma_t^\omega \in \mathrm{Aut}(\mathcal{A})$,
  - J_ω is the conjugate-linear operator spatially implementing the modular conjugation anti-isomorphism $\gamma_\omega : \pi_\omega(\mathcal{A}) \to \pi_\omega(\mathcal{A})'$,
  - $\mathcal{A}_\omega := \{a \in \mathcal{A} \mid [K_\omega, \pi_\omega(a)] \in \pi_\omega(\mathcal{A})''\}$,

[16] See section 6 in `arXiv:1007.4094v2`.

Modular Algebraic Quantum Theory 3

- Tomita-Takesaki modular theory is here taking the role of the quantum version of Einstein's equation associating "geometries" to "matter content" where:
  - "geometries" are spectrally described by variants of **modular spectral triples** (see A.Carey-A.Rennie-J.Phillips-F.Sukochev),
  - "matter content" is described by the set of quantum correlations between observables specified by the state.
- Every pair $(\mathcal{O}, \omega)$ gives a different "net" of modular spectral geometries $(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega)_{\mathcal{A}\subset\mathcal{O}}$ that are:
  - **quantum**, since $\mathcal{A} \subset \mathcal{O}$ are non-commutative,
  - **state-dependent** on ω,
  - **relative to observers** $\mathcal{O}$.

Modular Algebraic Quantum Theory 4: Looking Ahead

- every pair $(\mathcal{O}, \omega)$ selects C*-categorical data inside the C*-algebra $\mathcal{O}$: the family of algebras $\mathcal{A}$ and some of their "correlations bimodules";
- non-commutative space-time is now constructed topologically via the "C*-enveloping" of the base category and we guess that its spectral non-commutative geometry can be recovered from the additional spectral data of the modular spectral geometries living on the total space of such bundle;
- the investigation of "(higher) categorical modular theory" is now one of the priorities of this program :-)

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