Analytical continuation of Black Hole entropy in Loop Quantum Gravity

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Introduction

Purposes
- Defining a consistent analytical continuation of the entropy for black hole with the complex Ashtekar variables

Context
- Black hole in Loop Quantum gravity: treated as boundary in the spacetime (isolated horizon)
- Gravitational D.o.F on the horizon described by a Chern Simons theory living on this horizon with the constraint:
  \[ k = \frac{(1 - \gamma^2)A_h}{2\pi\gamma} \]
- For real Black Hole, \( \gamma \) plays a crucial role:
  \[ S_{bh} = \frac{A_h\gamma_0}{4\gamma} \quad \gamma = \gamma_0 \sim 0, 2375 \]
### Ideas

#### Problematic

- What happen if one want to work with $\gamma = i$?  
  [Frodden, Geiller, Noui, Perez (2012)]

#### Motivations

- $\gamma$ plays no role at the classical level
- Ashtekar-Barbero not a full space-time connection  
  [Samuel, Alexandrov]
- Three dimensional gravity : toy model  
  [Ben Achour, Geiller, Noui, Yu (2013)]
  - to solve the dynamic, need to work with an $\text{su}(1,1)$-connection
  - area spectrum becomes continuous and independent of $\gamma$

\[
A(s) = 8\pi l_p^2 \sqrt{s^2 + 1/4} \quad s \in \mathbb{R}^+
\]
Analytical continuation: first step

Problematic

What happen if one want to work with $\gamma = i$?
[Frodden, Geiller, Noui, Perez (2012)]

The Chern Simons level become purely imaginary

To keep an area real and positive, one need to send $k \rightarrow i\lambda$ when $\gamma = i$.

$$A_h = \frac{2\pi \gamma k}{1 - \gamma^2} \quad \gamma \in \mathbb{R} \quad \rightarrow \quad A_h = 2\pi \lambda \quad \gamma = i$$

Problem: defining an analytical continuation of the Chern-Simons theory for $k \rightarrow i\lambda$ [Witten (2010), Morse theory]
Analytical continuation: first step

Rewriting the Verlinde formula as an integral in the complex plane

Dimension of the Chern Simons Hilbert space $SU_q(2)$:

$$N_k(d_l) = \frac{2}{k+2} \sum_{d=1}^{k+1} \sin^2 \left( \frac{\pi d}{k+2} \right) \prod_{l=1}^{n} \frac{\sin \left( \frac{\pi}{k+2} d d_l \right)}{\sin \left( \frac{\pi}{k+2} d \right)}$$  

Can be reinterpreted as a sum of residues of the following integral:

$$I_k(d_l) = \frac{i}{\pi} \oint_C \sinh^2 z \prod_{l=1}^{n} \frac{\sinh(d d_l)}{\sinh(z)} \coth(k+2)z$$  

Poles in $z_p = \frac{i \pi p}{k+2}$ with $p \in \mathbb{N}^*$

Contour $C$ encloses the imaginary axis between $[0, i \pi]$
Analytical continuation : second step

Starting point

- Case \((d_l, k) \in \mathbb{N}\) : pole in \(z = \frac{i \pi p}{k+2}\) \(p \in \mathbb{N}\), on the imaginary axis

Naive analytical continuation : \(k \rightarrow i \lambda\)

- Case \(d_l \in \mathbb{N}\) \(k \in i \mathbb{R}\) : pole in \(z = -\frac{i \pi p}{\lambda}\) \(p \in \mathbb{N}^*\), on the real axis
- Integral vanish since the contour doesn’t enclose any pole : inconsistent

Consistent analytical continuation : \(k \rightarrow i \lambda\), \(d \rightarrow i s\)

- Case \(d_l \in i \mathbb{R}\) \(k \in i \mathbb{R}\) : pole in \(z = -\frac{i \pi p}{\lambda}\) \(p \in \mathbb{N}\), on the real axis and pole in \(z = i \pi m\) \(m \in \mathbb{N}^*\) on the imaginary axis
- Only way to have a nontrivial analytical continuation, make \(d\) purely imaginary : \(d = 2j + 1 \rightarrow d = i s\), \(j \rightarrow -\frac{1}{2} + is\), \(s \in \mathbb{R}^+\)
Thermodynamical study

Definition of the number of microstates

- $\gamma = i$, $k = i\lambda$, $d = is$
  
  For large black hole, $\lambda \gg 1$

\[
I_\infty(s, n) = \frac{i}{\pi} \oint_C \sinh^2(z) \prod_{l=1}^n \frac{\sinh(sz)}{\sinh(z)} \quad s \in \mathbb{R}
\]

One color model

- Same color for each puncture: one color model

\[
I_\infty(s, n) = \frac{i}{\pi} \oint_C \sinh^2(z) e^{nS(z)} \quad S(z) = \log \frac{\sinh(sz)}{\sinh(z)}
\]

- The thermodynamical limit: $n \gg 1$, stationnary phase method
### Partition function and entropy

#### Microcanonical ensemble

- Entropy (with a Gibbs factor)

\[
S = \frac{a_H}{4 l_p^2} + B \sqrt{\frac{a_H}{2\pi l_p^2}} - 2\log\left(\frac{a_H}{l_p^2}\right)
\]

#### Grand canonical ensemble

- Partition function
  - Indistinguishability of the punctures [Gosh, Noui, Perez (2014)]

\[
Z = \int ds \sum \frac{1}{n!} g(n, s) e^{-\beta E}
\]

- Frodden-Gosh-Perez notion of energy: \( E = \frac{A}{8\pi L} \).

\[
Z(\beta) \sim \sqrt{8\pi x^3} \exp\left(\frac{1}{2x}\right) \quad x = \frac{l_p^2}{2L} (\beta - \beta_U)
\]
Grand canonical ensemble

Entropy with $\mu = 0$

- Mean area

$$< a_H > = \frac{2\pi l_p^2}{x^2}$$

- Thermodynamical limit: $x \to 0$.

- Mean number of puncture and mean color

$$< n > = \frac{1}{8\pi} \sqrt{< a_H >}$$

$$< s > = \frac{1}{4\pi} \sqrt{< a_H >}$$

- Semiclassical regime dominated by large spins!

- Entropy

$$S = \frac{a_H}{4l_p^2} + \sqrt{\frac{a_H}{2\pi l_p^2}} - \frac{3}{2} \log\left(\frac{a_H}{l_p^2}\right)$$

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Grand canonical system

Entropy when $\mu = 2T_U$

- Non zero chemical potential for the punctures: $\mu = 2T_U$
- Entropy:

$$S = \frac{a_H}{4l_p^2} - \frac{3}{2} \log(\frac{a_H}{l_p^2})$$

- Non vanishing chemical potential / black hole release energy when one remove a puncture = radiation

Main result

- With $\gamma = i$, the Bekenstein Hawking area law is recovered without any unnatural fine tuning + logarithmic corrections
Conclusion and perspectives

General perspectives for black hole physics

- Define a unique and consistent analytical continuation of black hole entropy
- Introducing quantum statistic for the punctures gas

General perspectives for LQG

- Give strong indications how to work on the self dual side of the theory
  Precise prescription for the analytical continuation
- Applying the same prescription to the real LQC [Ben Achour, Grain, Noui (2014) arXiv:1407.3768 [gr-qc]]
- Could give new ideas to resolve the reality conditions, [Thiemann, Ashtekar (1995)]
- Give new insight on the status of the Barbero Immirizi parameter: regulator to be send to $i$ to get physical predictions