Background-independent renormalization in quantum gravity models

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• Motivation: Why renormalization in spin foam models?
• Renormalization without a length scale
• Diffeomorphism symmetry
• Easy examples
• How to do it in practice: Approximation methods
• Summary
EPRL Spin Foam model

\[ Z_{\text{EPRL}} = \sum_{k_f, l_e} \prod_{e} A_{e} \prod_{f} A_{f} \prod_{v} A_{v} \left( \prod_{e, \partial} B_{e} \prod_{v, \partial} B_{v} \right) \]

boundary terms

Triangulation (originally embedded, nowadays mostly abstract)

Geometrical interpretation of boundary states as discrete (twisted) geometry

Physical input: implementation of simplicity constraints

\[ B = \frac{1}{\gamma} (e \wedge e) + \frac{1}{\gamma} e \wedge e \]

onto state-sum model for 4d BF theory

[Reisenberger '94, Barrett, Crane '99, Livine, Speziale '07, Engle, Pereira, Rovelli, Livine '07, Freidel, Krasnov '07, Kamiński, Kisielowski, Lewandowski '09, Han, Thiemann '10, Oriti Baratin '11, ...]

[Plebański '77, Capovilla, Jacobson, Dell, Mason '91]

[Ponzano, Regge, '69, Horowitz '89, Baez '99]
Renormalization á la Wilson

General feature of physical systems:

\[ \infty \text{ many d.o.f. } + \text{ nonlinearity } = \text{ „running of coupling constants“} \]

d.o.f. ordered along scales – dynamics is different at different scales

[crf Wilson '71]
Classical General Relativity

Gravity $\leftrightarrow$ Curvature of space-time metric $g_{\mu\nu}$

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \ R$$

Diffeomorphisms: $\phi : \mathcal{M} \rightarrow \mathcal{M}$

$$g_{\mu\nu}(x) \rightarrow \frac{\partial \phi'\mu}{\partial x\mu} \frac{\partial \phi'\nu}{\partial x\nu} g_{\mu'\nu'}(\phi(x))$$

$$T_{\mu\nu}(x) \rightarrow \frac{\partial \phi'\mu}{\partial x\mu} \frac{\partial \phi'\nu}{\partial x\nu} T_{\mu'\nu'}(\phi(x))$$

Einstein's "hole argument":

\[ \text{diffeomorphisms} = \text{gauge symmetry of GR} \]

So what are the scales in Spin Foams?
Two different philosophies

1.) EPRL "fundamental":

Physical triangulation with EPRL as fundamental theory

Then what is continuum limit?

2.) GR "emergent"

Triangulation = technical tool
Microscopic theory yet unknown

Discreteness of space-time unclear!

either way: RG flow will be nontrivial, has to be computed!

[Vidotto, Rovelli '09]
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Generalized spin foam models

Start with manifold $\mathcal{M}$ and consider embedded, oriented 2-complexes $\Gamma \subset \mathcal{M}$ variables: group elements $h_{ef}$ for $e \subset f$

$h_{ef} \in G$ compact Lie group

configuration space: $\mathcal{A}_\Gamma \simeq G^n$

integration measure: $\mu_\Gamma$

observables: $\mathcal{O}_\Gamma \in C^0(\mathcal{A}_\Gamma)$

$$\langle \mathcal{O} \rangle_\Gamma = \int_{G^n} d\mu_\Gamma(h_{ef}) \mathcal{O}(h_{ef})$$

[Pfeiffer ‘01, Pfeiffer, Oeckl ’02, Kamiński, Kisielowski, Lewandowski ’09, Magliaro, Perini, ’10, BB, Dittrich, Hellmann, Kamiński ’12]
Generalized spin foam models

special cases: lattice gauge theory:

\[ G = SU(N), \Gamma \quad \text{hypercubic lattice} \]

\[ d\mu_\Gamma(h_{ef}) = dh_{ef} \prod_f e^{-S_{\text{Wilson}}[H_f]} \prod_e \int_G dg_e \prod_{f \subset e} \delta(g_e, h_{ef}) \]

EPRL spin foam model:

\[ G = SU(2) \times SU(2), \Gamma \quad \text{dual to 4d triangulation} \]

\[ d\mu_\Gamma(h_{ef}) = dh_{ef} \prod_f \delta(H_f) \int dg_{ve} \prod_{ef} E(g_{ve}^{-1}h_{ef}g_{we}) \]

choice of measure \( \mu_\Gamma \Leftrightarrow \) choice of theory
Continuum limit

Natural partial ordering of \( \Gamma \)'s (in semi-analytic category)

\[ \Gamma \leq \Gamma' \]

Not every two \( \Gamma \)'s can be compared, but: for each two \( \Gamma, \Gamma' \) there is a finer one:

projection ("coarse graining map")

\[ \pi_{\Gamma'\Gamma} : \mathcal{A}_{\Gamma'} \rightarrow \mathcal{A}_{\Gamma} \]

\[ \pi_{\Gamma'\Gamma}([h_{e'f'}])_{ef} = \prod_{e' \subseteq e, f' \subseteq f} h_{e'f'} \]
Continuum limit

Consider all $\Gamma$ (or sufficiently many) at the same time:

projective limit: \[ \overline{A} := \lim_{\Gamma} A_{\Gamma} = \{ \{ a_{\Gamma} \}_{\Gamma} \mid a_{\Gamma} \in A_{\Gamma}, \pi_{\Gamma',\Gamma} a_{\Gamma'} = a_{\Gamma} \} \]

space of (generalized) continuum connections: compact Hausdorff space \[ A \subset \overline{A} \]

condition for continuum measure $\mu$ on $\overline{A}$:

\[ (\pi_{\Gamma',\Gamma})_* \mu_{\Gamma'} = \mu_{\Gamma} \]

„cylindrical consistency“

\[ \{ \mu_{\Gamma} \}_{\Gamma} \to \mu \quad \text{Radon measure on } \overline{A} \]
### Dictionary

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\overline{\mathcal{A}}$</td>
<td>quantum continuum connections</td>
</tr>
<tr>
<td>$\mathcal{A}_\Gamma$</td>
<td>are finite-dim „slices“ through $\overline{\mathcal{A}}$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>provides cut-off: only finitely many holonomies</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>The two-complexes are the scales!</td>
</tr>
<tr>
<td>$\mu$</td>
<td>continuum path integral measure („full theory“)</td>
</tr>
<tr>
<td>$\mu_\Gamma$</td>
<td>partial path integral measure („effective theory“)</td>
</tr>
<tr>
<td>$\mathcal{O}_\Gamma$</td>
<td>observables at scale $\Gamma$</td>
</tr>
<tr>
<td>$\Rightarrow$</td>
<td>only finitely many holonomies measurable at a time</td>
</tr>
</tbody>
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<tr>
<td>$\langle \mathcal{O}_\Gamma \rangle$</td>
<td>expectation values of observables:</td>
</tr>
<tr>
<td>$\pi_{\Gamma'\Gamma}$</td>
<td>coarse graining from scale $\Gamma'$ to scale $\Gamma$</td>
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Dictionary ctd

cylindrical consistency: \((\pi_{\Gamma',\Gamma})_*\mu_{\Gamma'} = \mu_{\Gamma}\)

is precisely the idea of Wilsonian RG flow!

\[
d\mu_{\Gamma} = e^{-S[h_e, g_i(a) \mid a]} \, dh_e
\]

measure ("action") parametrised by parameters \(g_i(a)\) which depend on \(a\)

\[
e^{-S[h_e, g_i(a)]} = \int dh_{e'} \, e^{-S[h_{e'}, g_i(a')] \mid a'} \delta \left( h_e = \prod_{e' \subseteq e} h_{e'} \right)
\]
"scale" with background:

\( g_i(a) \)

convergence in the sense of sequence:

\( a \)

\( a' = \frac{a}{2} \)

"scale" without background:

\( \{ \mu_\Gamma \}_\Gamma \)

convergence in sense of filters:

\( \Gamma^{(1)} \)

\( \Gamma^{(2)} \)

\( \Gamma^{(3)} \)
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Diffeomorphism group action

Because one considered embedded $\Gamma$

Diffeomorphism action $\phi : \mathcal{M} \rightarrow \mathcal{M}$
Acts on continuum configurations:

$$\phi : \overline{A} \rightarrow \overline{A}$$

invariance of continuum measure under $\phi$

$$\Leftrightarrow \quad \phi_* \mu = \mu$$

equivalent to: $\langle \mathcal{O} \rangle_\Gamma = \langle \phi^* \mathcal{O} \rangle_{\phi(\Gamma)}$ for all $\Gamma$ supporting $\mathcal{O}$
**Diffeomorphism group action**

Diffeomorphism-invariance of partial measures $\mu_\Gamma$

$$\langle \mathcal{O} \rangle_\Gamma = \langle \phi^* \mathcal{O} \rangle_{\phi(\Gamma)}$$

Together with cylindrical consistency this is a very strong condition:

$$\mathcal{O}_1 = \mathcal{O} \circ \pi_{\Gamma'\Gamma}$$
$$\mathcal{O}_2 = \phi(\mathcal{O}) \circ \pi_{\Gamma'\phi(\Gamma)}$$

$$\langle \mathcal{O}_1 \rangle_{\Gamma'} \overset{1}{=} \langle \mathcal{O}_2 \rangle_{\Gamma'}$$

see talk by Etera Livine!
Hamiltonian formulation

Manifold with boundary, e.g.
\[ \partial \mathcal{M} = \overline{\Sigma_1} \sqcup \Sigma_2 \]
boundary graph \[ \partial \Gamma = \overline{\gamma_1} \sqcup \gamma_2 \]
boundary holonomies \[ \mathcal{B}_\gamma = G^m_\gamma \]
kinematical boundary Hilbert space:
\[ \mathcal{H}_\gamma = L^2(\mathcal{B}_\gamma) \]
boundary observables: \[ \psi \in C^0(\mathcal{B}_\gamma) \]
provides physical inner product in sense of rigging map in RAQ

\[ \langle \psi_1 \mid \psi_2 \rangle_{\Gamma,\text{phys}} := \langle \overline{\psi_1} \otimes \psi_2 \rangle_{\Gamma} \]
cylindrical consistency of \[ \langle \cdot \rangle_{\Gamma} \] guarantees extension to continuum boundary HS

\[ \mathcal{H}_\Sigma = \lim_{\gamma \to \infty} \mathcal{H}_\gamma \]
LQG kinematical Hilbert space (for EPRL model)!

[Kamiński, Kisielowski, Lewandowski, Puchta '11, Bahr, Hellmann, Kamiński, Lewandowski, '12]
[Ashtekar et al '95, Marolf, Guillini '98, Thiemann '01]
[Kamiński, Kisielowski, Lewandowski] '09
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Example:

Near trivial example: \[ \dim \mathcal{M} = 2, \quad G = U(1) \quad \mathcal{A}_\Gamma = U(1)^E \]

„charge-network functions“ \[ \mathcal{O}(h_e) = \prod_e h_e^{m_e} \]

\[
\langle \mathcal{O} \rangle_\Gamma := \frac{1}{Z} \int_{U(1)^E} dh_e \mathcal{O}(h_e) \prod_f \sum_{n_f \in \mathbb{Z}} \exp \left( -n_f^2 a_f / 2 + i \theta_f n_f \right) \left( \prod_{e \in f} h_e^{[e,f]} \right)^{n_f} \\
= \int_{U(1)^E} d\mu_{\Gamma,\overline{\Gamma}}(h_e) \mathcal{O}(h_e) \quad a_f > 0, \quad \theta_f \in \mathbb{R}
\]

coarse graining map:

\[
\pi_{\Gamma' \Gamma}(h_e) = (\cdots, h_{e_1} h_{e_2} h_{e_3}^{-1}, \cdots) = h_{e'}
\]
Example:

RG equations:

\[
\alpha_f^\Gamma = \sum_{f' \subset f} \alpha_{f'}^\Gamma
\]
\[
\theta_f^\Gamma = \sum_{f' \subset f} [f', f] \theta_{f'}^\Gamma
\]

Obvious solution:

\[
g \in \text{Sym}^2 T^* M \quad \text{(area) metric}
\]
\[
\theta \in \Omega^2(M) \quad \text{2-form}
\]

\[
\alpha_f^\Gamma = \int_f \text{dvol}_g \quad \theta_f^\Gamma = \int_f \theta
\]

Limit solutions:

\[
\theta = 0 \quad \text{2d Yang-Mills on background metric } g
\]

\[
g \to 0 \quad \int_A DA \delta(F[A] - \theta)
\]
Example:

Diffeomorphism-invariance:

\[
\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\Gamma} = \langle \phi_* \mathcal{O} \rangle_{\phi(\Gamma)} = \langle \phi_* \mathcal{O} \rangle
\]

\[
\Rightarrow \quad \langle \mathcal{O} \circ \pi_{\Gamma'} \Gamma' \rangle_{\Gamma'} = \langle \phi_* \mathcal{O} \circ \pi_{\Gamma'} \phi(\Gamma) \rangle_{\Gamma'}
\]

\[
\Rightarrow \quad a_{f_1} + a_f + a_{f_2} = a_{f_1} - a_f + a_{f_2} \quad \theta_{f_1} + \theta_f + \theta_{f_2} = \theta_{f_1} - \theta_f + \theta_{f_2}
\]

Only solutions: \( g \to 0, \infty \) \quad \theta = 0

(limits exist as measures on \( \overline{\mathcal{A}} \) )

Could have been guessed from \( \phi_* \mu^{(g,\theta)} = \mu^{(\phi_* g, \phi_* \theta)} \)
Example:

Space of solutions to GR equations:

\[
d\mu^{(g,\theta)} = \frac{1}{Z} \prod_e dh_e \prod_f \sum_{n_f \in \mathbb{Z}} \exp \left( -n_f^2 a_f / 2 + i\theta_f n_f \right) \left( \prod_{e \subset f} h_e^{[e,f]} \right)^{n_f}
\]

\[
a_f = \int_f d\text{vol}_g \quad \theta_f = \int_f \theta
\]
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Summary
Approximations:

RG flow equations: conditions of cylindrical consistency of partial measures

\[ \langle \mathcal{O} \rangle_{\Gamma} = \langle \mathcal{O} \circ \pi_{\Gamma' \Gamma} \rangle_{\Gamma'} \]

for all \( \Gamma \leq \Gamma' \) and all \( \mathcal{O} \in C^0(A_{\Gamma}) \)

\[ \Rightarrow \text{In most cases impossible to check.} \]

Possible approximations:

- Don't check for all 2-complexes \( \Gamma \), just for subset (e.g. lattices, or hose dual to triangulations)

- Don't check on all observables \( \mathcal{O} \), just on few interesting ones

- Search for solutions in subset of all measures, e.g.

\[ d\mu(h_{ef}) = \prod_{ef} dh_{ef} \exp \left( -S^{(g_i)}[h_{ef}] \right) \]

truncation to few parameters \( g_i \) and minimize

\[ \sum_n c_n \| \langle \mathcal{O}_n \rangle_{\tilde{g}} - \langle \mathcal{O}_n \circ \pi_{\Gamma' \Gamma} \rangle_{\tilde{g}'} \|^2 \]
Example 1:

Use finite group:

\[ G = S_3 = \{1, (12), (13), (23), (123), (132)\} \]

has three irreps: trivial \(0_{S_3}\), sign \(1_{S_3}\) and \(2_{S_3}\)

\[
\rho_{2_{S_3}}(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_{2_{S_3}}(123) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}
\]

\(c = \cos(2\pi/3), \quad s = \sin(2\pi/3)\)

Truncation of measure space (mimics EPRL model):

\[
d\mu_\Gamma = dh_{ef}\prod_f \delta(H_f) \int dg_{ve} \prod_{e \subset f} \left( e_0 + e_1 \rho_{1_{S_3}}(\tilde{h}_{ef}) + e_{01} \rho_{1_{S_3}}(\tilde{h}_{ef})^{\uparrow\uparrow} + e_{02} \rho_{1_{S_3}}(\tilde{h}_{ef})^{\downarrow\downarrow} \right)
\]

\[\tilde{h}_{ef} = g_{ve}^{-1} h_{ef} g_{we}\]
Example 1:

3d: hierarchical lattics made from tetrahedra, only consider observables at boundary
RG step:
Example 1:

Numerical flow:

I: $S_3$ BF theory
II: AL measure
III: $\mathbb{Z}_2$ BF theory
IV: nontrivial
Example 2:

\[ \dim \mathcal{M} = 2, \quad G = U(1) \quad \mathcal{A}_\Gamma = U(1)^E \]

\[ d\mu^{(t,\lambda)} := \frac{1}{Z} d^E h_e \prod_f \mathcal{K}_f(H_f) \exp \left( -\lambda \sin^4(H_f) \right) \]

\[ t > 0, \ \lambda \in \mathbb{R} \]
Example 2:
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Summary:

- Quantum GR is definitely nonlinear and is a field theory
  ⇒ Renormalization (in Wilsonian sense) needs to be understood in spin foam models.

- Scales are a way of creating a hierarchy of d.o.f.
  ⇒ embedded 2-complexes provide a natural hierarchy
  ⇒ RG flow along poset, rather than „length scale“

- RG equations ⇔ cylindrical consistency from measure theory

- Although setup is background-independent, a solution not necessarily is!
  ⇒ Diffeomorphism-invariance strong condition!

- Approximation methods are being developed, and yield first numerical results
  crf work by Riello, Rovelli for analytical results!
Open questions:

- How to incorporate Lorentzian signature? (non-compact groups)
- Other interesting diff-invariant models (lower-dimensional, Seiberg-Witten?)
- How good are approximation methods (e.g. compared to Migdal-Kadanoff)?
- Interesting coupling constants for QGR: R-terms, etc.
- Connection to canonical formulation: constraints from RG flow?
- Asymptotic safety? Renormalizability?
Naïve hope for the future:

EPRL (maybe with $\alpha R^2$-corrections?), depending on few parameters $\kappa, \gamma, \Lambda, \alpha$

Form invariance: NGFP + no nonrenormalizability

crf talks by Saueressig, Litim!