How Many Quanta are there in a Quantum Spacetime?

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What I'm going to talk about..

- *"Given a chunk of space as a slice of spacetime, how many quanta does it contains?"*. This question is ill-posed. **Why?**
- Anything else? Coarse-graining for a system of quanta of space.

Background and motivation

Why asking such question?

- Important for counting state for blackholes, thermodynamics aspect of LQG, etc.
- Need to clarify things: there is confusion when people talk about quanta.
- Quanta are *not* defined globally, it depends on what we want to measure.

Outline

1. What is a particle?

What is a 'particle'?

- **Classical Physics:** *"..entity with mass, may have volume, localized in space, have a well-defined boundary."*
- **Quantum Mechanics:** 'Quanta' of energy.
- **Quantum Field Theory:** 'Quanta' of energy from the excitation of the field.
- Notion of 'particles' *depends on coordinates / basis chosen.*

'Particles' depends on coordinates: Quantum mechanics example.

- System of 2 uncoupled harmonic oscillator can be written in different coords:

vars.	Hilbert space	State	# ops.
$\{(q_1, p_1), (q_2, p_2)\}$	$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$	$\|n_1, n_2\rangle$	$\hat{N}_{12}$
$\{(q_{\mathrm{CM}}, p_{\mathrm{CM}}), (q_{\mathrm{r}}, p_{\mathrm{r}})\}$,	$\mathcal{H} = \mathcal{H}_{\mathrm{CM}} \otimes \mathcal{H}_{\mathrm{r}}$	$\|n_{\mathrm{CM}}, n_{\mathrm{r}}\rangle$	$\hat{N}_C$
$\{(q_+, p_+), (q_+, p_+)\}$	$\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$	$\|n_+, n_-\rangle$	$\hat{N}_\pm$

- Have same Lagrangian and Hamiltonian.
- Acting the number operators on the state, $|\psi\rangle$ expanded in different basis will give different number of quanta: $n_1 + n_2$, $n_{\mathrm{CM}} + n_{\mathrm{r}}$, $n_+ + n_-$.

What is a particle?

Particles and number of particles depend on the coordinate / basis chosen.

2. Quanta of space

Quanta of space.

Quanta of space is..

- ..a quanta of energy from the excitation of the gravitational field.
- In loop quantum gravity, each quanta is a 'quantum polyhedron'.
- The geometry of quantum polyhedron defined by graph.
- We associate state (element of Hilbert space) for quanta of space.
- The basis which spanned this Hilbert space is the spin network basis.

Spin network state in LQG.

- Spin network basis: $|j_l, i_n\rangle$ or $|j_l, v_n\rangle$.
- It diagonalized the area and the volume of the tetrahedron.
- Area operator is $A_{nn'} = 8\pi\gamma G|J_{nn'}|$, and the volume is $v(J_{nn'})$.

Quanta of space

Space in LQG is discretized by a quanta of space, the state of space is expanded using spin network basis.

3. Transformation of spin network basis

Transformation of spin network basis

- We want to have a spin network analog to the transformation-to-center-of-mass-coord.

$$|x_1, x_2\rangle \Longleftrightarrow |x_{\mathrm{CM}}, x_{\mathrm{r}}\rangle ,$$

  differs in the 'size of grains'.
- In analog:

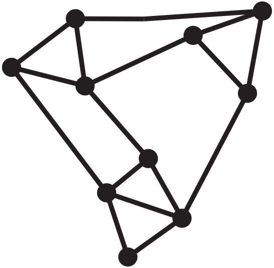
$$|j_l, v_n\rangle \Longleftrightarrow |j_L, v_N, \alpha\rangle ,$$

  j_L, v_N is the 'center-of-mass' or 'big grains' quantum numbers, α is the 'reduced' quantum number.
- How to define 'big grains' in spin network? $\rightarrow$ arbitrary division of graph into subgraph $\rightarrow$ subset graph.

Subset Graph: definition

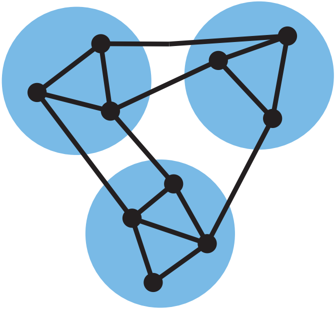
- Earlier studies about the relation between graph: Livine and Terno [2],
- Given a graph γ, we define "subset graph" Γ as follow:

Subset Graph: definition



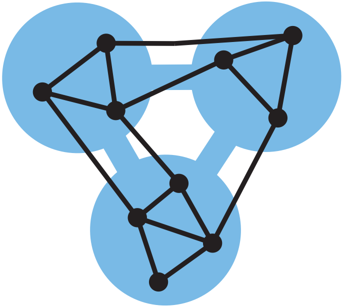
- Given a graph γ

Subset Graph: definition



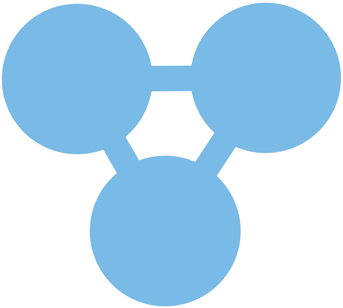
- Consider a partition of $\mathcal{N}$ into subsets $N = \{n, n', n'', ...\}$, called "big nodes", such that N is a connected component of γ.

Subset Graph: definition



- Consider two such big nodes N and N'. They are "connected" if there is at least one link of γ that links a node in N with a node in N', then there is a "big link" $L = (N, N')$ connecting the two.

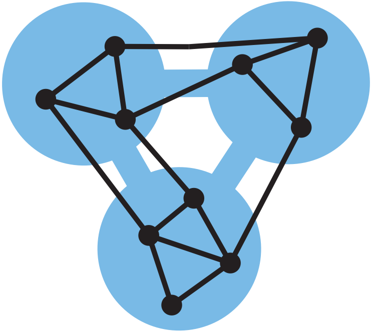
Subset Graph: definition



- The set of the big nodes and the big links defines a graph, which we call "subset graph" Γ of γ. .

Subset Graph: definition

Together:



A graph γ in black and subset graph Γ in blue.

The algebra and holonomy of the subset graph

- Algebra of operators and holonomies in $\mathcal{H}_\Gamma$, of subset graph Γ for each big link L:

$$\vec{J}_L := \sum_{l \in L} \vec{J}_l$$

$$U_L := U_l$$

- the algebra structure $\mathcal{J}_\Gamma$ of variables in Γ is

$$\begin{aligned} \left[J^i_L, J^j_{L'}\right] &= \delta_{LL'} \epsilon^{ij}{}_k J^k_L \\ \left[J^i_L, U_{L'}\right] &= \delta_{LL'} \tau^i U_L, \\ \left[U_L, U_{L'}\right] &= 0 \end{aligned}$$

the same structure with $\mathcal{J}_\gamma$ in graph γ!

Spin network state of subset graph

- Subset graph Γ of γ is a well-defined graph.
- Can obtain the state in the same way as before, by using spin network basis $|j_L, v_N\rangle$.
- The non-gauge invariant Hilbert space is $\mathcal{H}_\Gamma \subsetneq \mathcal{H}_\gamma$
- Taking gauge invariant, we obtain the invariant subspace: $\mathcal{K}_\Gamma \subsetneq \mathcal{H}_\Gamma$
- Now with these definition, we can start to write the transformation we want, precisely!

Transformation of spin network basis

- **Assumption:** For every $\mathcal{K}_\gamma \subseteq \mathcal{H}_\gamma$ and $\mathcal{K}_\Gamma \subseteq \mathcal{H}_\Gamma$, there exist transformation:

$$\begin{aligned} \Lambda : \mathcal{K}_\gamma &\rightarrow \mathcal{K}_\gamma \\ |j_l, v_n\rangle &\rightarrow |j_L, v_N, \alpha\rangle , \end{aligned}$$

  where $|j_L, v_N\rangle \in \mathcal{K}_\Gamma$.
- As a consequence of this assumption, $\mathcal{K}_\Gamma$ must be inside $\mathcal{K}_\gamma$: $\mathcal{K}_\Gamma \subseteq \mathcal{K}_\gamma$.
- But there is a problem: In general, it is not: $\mathcal{K}_\Gamma \nsubseteq \mathcal{K}_\gamma$. Even worse, there is a case where:

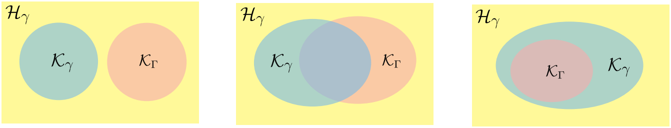
$$\mathcal{K}_\Gamma \cap \mathcal{K}_\gamma = \varnothing$$

Problem

- Fact:

$$\mathcal{K}_\gamma \subsetneq \mathcal{H}_\gamma, \quad \mathcal{H}_\Gamma \subsetneq \mathcal{H}_\gamma, \quad \mathcal{K}_\Gamma \subsetneq \mathcal{H}_\Gamma, \quad \mathcal{K}_\gamma \cap \mathcal{H}_\Gamma \neq \varnothing$$

- Is $\mathcal{H}_\Gamma \subseteq \mathcal{K}_\gamma$? It is 'no' in general. (it is, in a 'flat' case).
- Diagram of all possible case as follow:



three possibilities of the relation between $\mathcal{K}_\gamma$ and $\mathcal{K}_\Gamma$

- Using the same simple example, we can show that case $\mathcal{K}_\Gamma \cap \mathcal{K}_\gamma = \varnothing$ exist.

Problem

- So, in general, $\mathcal{K}_\Gamma \nsubseteq \mathcal{K}_\gamma$.
- Problem: don't have transformation Λ we assumed before: don't have an analog of transformation-to-center-of-mass coord for spin network states.
- But instead, we propose another map, another procedure: coarse-graining!

Transformation of spin network basis

There is no analog of 'basis-transformation' which differs in the size of grains inside physical Hilbert space for spin network, so we propose coarse-graining map as an alternative.

4. Coarse-graining

What is coarse-graining?

- .. is a procedure to describe physicals system using smaller number of variables, fewer degrees of freedom.
- Example: the motion of a many-body problems is described by physics of the center of mass, which coarse-grains the variables of the individual bodies.
- Coarse-grained observables are quantized and can be discrete as a consequence of quantum theory.
- In terms of graph, coarse-graining is deforming the graph by reducing number of links and / or node.
- The Hilbert space of the coarse-grained graph is smaller than the fine grained graph : degrees of freedom reduced.

Coarse-graining map

Why do we need coarse-graining procedure? What does coarse-graining has to do with the question?

- There is no transformation Λ in general: no transformation from $\mathcal{K}_\gamma$ to $\mathcal{K}_\gamma$ which have different number of node in the graph.
- To still have a 'transformation' of spin network basis, we define the coarse-graining map:

$$\pi_{\gamma\Gamma} : \mathcal{K}_\gamma \otimes \mathcal{K}_\gamma^* \to \mathcal{K}_\Gamma \otimes \mathcal{K}_\Gamma^*$$

- Instead of using a pure state, we use a more general mixed state: density matrix.

Coarse-graining map

- How does the map works?
- How to obtain coarse-grained density matrix from the fine grained state?
- The whole process is complicated, so we will only explain the general idea using an example.

Sketch of the general idea

- Take a 6 links example . Work in non-gauge full Hilbert space, and use $|j, m, n\rangle$ basis.
- We know that

$$\otimes^{l}\mathcal{H}^{j_l} = \mathcal{H}^{j_{\min}} \oplus \ldots \oplus \mathcal{H}^{j_{\max}}, \quad j_{\min} \geq 0.$$

- Then  could be transformed to  and then to .
- we see  is the subset graph we want, so we need to throw  away, by tracing out!
- This is analog of treating the system as a single body problem, we don't need information about the 'reduced' variables: tracing away: losing information of 'reduced' variables.

Sketch of the general idea

- This is why density matrix enters, because in general, tracing out degrees of freedom from a Hilbert space in QM will correspond to mixed state:

$$\rho_\Gamma = \mathrm{tr}_{\mathcal{H}_\equiv} |\Rrightarrow\rangle |\curlywedge\rangle\langle \curlyvee| \langle \Lleftarrow|$$

- ρ_Γ is the coarse-grained density matrix we want, in general it will be mixed, due to the statistical uncertainty of the states. Lose degrees of freedom in this step.
- ρ_Γ is still a non gauge invariant state. Take the gauge invariant part $J = 0$. Also lose more degrees of freedom in this step.
- This is what we are going to do to spin network states!

Physical state: Taking the gauge invariant subspace

- Gauge invariance acts differently on different graph: because they have different number of nodes and different configuration of the links!
- Project the density matrices $\rho_\gamma \in \mathcal{H}_\gamma \otimes \mathcal{H}_\gamma^*$ and $\rho_\Gamma \in \mathcal{H}_\Gamma \otimes \mathcal{H}_\Gamma^*$ using a projector π_n and π_N on each nodes n and N of the graph γ and graph Γ, respectively:

$$\rho_\gamma^{(\mathrm{inv})} = \pi_n \rho_\gamma \pi_n, \quad \pi_n = \int_{SU(2)} d\lambda_n \; \lambda_n$$

$$\rho_\Gamma^{(\mathrm{inv})} = \pi_N \rho_\Gamma \pi_N, \quad \pi_N = \int_{SU(2)} d\lambda_N \; \lambda_N$$

Finally, coarse-graining map!

- The map

$$\begin{aligned} \pi_{\gamma\Gamma} : \mathcal{K}_\gamma \otimes \mathcal{K}_\gamma^* &\rightarrow \mathcal{K}_\Gamma \otimes \mathcal{K}_\Gamma^* \\ \rho_\gamma^{(\mathrm{inv})} &\rightarrow \rho_\Gamma^{(\mathrm{inv})} \end{aligned}$$

  with Γ is a subset graph of γ , is the coarse-graining map.
- We're done!

Coarse-graining.

..is proposed because we don't have a well-defined basis transformation analog to the center of mass transformation inside the gauge invariant subspace.

5. Geometrical interpretation

Coarse grained area, coarse grained volume

- We have area and volume operator for γ : $A_l = 8\pi\gamma G|J_l|$, and the volume is V_n .
- and also for Γ: $A_L = 8\pi\gamma G|J_L|$, and the volume is V_N .
- We called them as coarse-grained area and coarse grained volume.
- The coarse-graining map $\pi_{\gamma\Gamma}$ sends state $\rho_\gamma^{(\mathrm{inv})}$ to a state with different curvature $\rho_\Gamma^{(\mathrm{inv})}$, because the gauge invariance acts differently on different node, it guarantees the 'flatness' on each node.

'Decomposition' of graph γ.

- Consider a family of graphs γ_m, with $m = 0, 1, .., M$ such that γ_{m-1} is a subset graph of γ_m and $\gamma_M = \gamma$.
- 'decomposition' of γ. γ is the finest graph in the family.
- The non-gauge invariant Hilbert space are nested into one another, there is a projection:

$$\pi_{\gamma_m \gamma_{m-1}} : \mathcal{H}_{\gamma_m} \otimes \mathcal{H}^*_{\gamma_m} \rightarrow \mathcal{H}_{\gamma_{m-1}} \otimes \mathcal{H}^*_{\gamma_{m-1}}$$

for each $m > 0$.

Full sets of different basis for a space

- We have a set of Hilbert spaces corresponding to a collection of subsets graph, nested into one another.
- But it doesn't mean the invariant subspaces also nested into one another.

Full sets of different basis for a space

- The set of area and volume operators A_L^m and V_N^m on each $\mathcal{H}_{\gamma_m}$ give a coarse grained description of the geometry, which becomes finer as m increases.

Full sets of different basis for a space

- Nice and famous example: the fractal of the Great Britain coastline!

Fock space for spin network

- Fock space in QFT:

$$\mathcal{F} = C \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \dots$$

- Fock space for spin network:

$$\mathcal{F} = C \oplus \mathcal{H}_{\gamma_0} \oplus \mathcal{H}_{\gamma_1} \oplus \dots \oplus \mathcal{H}_{\gamma_{m-1}} \oplus \mathcal{H}_{\gamma_m} \oplus \dots \oplus \mathcal{H}_{\gamma_M}$$

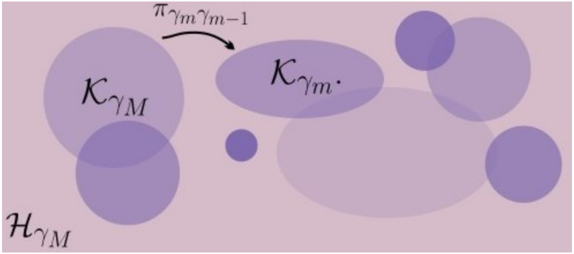
- The finest graph $\gamma_M = \gamma$ gives the truncation we want for the theory, the Hilbert space stops at $\mathcal{H}_{\gamma_M}$.

Comparison with QFT

- QFT: the Hilbert space of the system is fixed: $\mathcal{H}$, we can use any basis to expand it, there are transformation Λ between basis.
- In spin network, the Hilbert space of the truncated degrees of freedom is fixed: $\mathcal{H}_{\gamma_M}$. We take the invariant subspace $\mathcal{K}_{\gamma_M}$. But in general there is no transformation Λ inside $\mathcal{K}_{\gamma_M}$. Instead we have coarse-graining map $\pi_{\gamma_m \gamma_{m-1}}$ which maps different invariant subspaces inside $\mathcal{H}_{\gamma_M}$. Different invariant subspaces $\mathcal{K}_{\gamma_m}$ correspond to different basis, different graph, different number of quanta.

Comparison with QFT

- This is how we do 'basis-transformation' (in the sense it differs by the size of grains) in LQG: by moving from invariant subspace to invariant subspace!



Conclusion

Back to the question: How many?

Why is question of the title ill-posed?

- Measuring $\Longleftrightarrow$ interacting.
- How we interact with the system is realized by the choice of basis to span the Hilbert space of the system.
- Different choice of basis $\Longleftrightarrow$ to different observables to be measured.
- To obtain number of quanta in the system, we must give further information about which kind of quanta that we want: in what basis is the Hilbert space.
- Measuring different number of quanta on a same system $\Longleftrightarrow$ measuring the system using different 'resolution'.
- The observable we are measuring could be described by quantum numbers of coarse-grained operator, not the maximally fine grained ones.

Thank you!

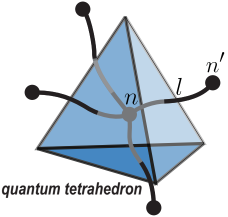
References.

1. D. Colosi and C. Rovelli, "What is a particle?" Class. Quant. Grav. 26 (2009) 25002,
2. E. R. Livine and D. R. Terno, "Reconstructing Quantum Geometry from Quantum Information: Area Renormalization, Coarse-Graining and Entanglement on Spin-Networks",
3. S. Ariwahjoedi, J.S. Kosasih, C. Rovelli, F. P. Zen. "How many quanta are there in a quantum spacetime?", http://arxiv.org/abs/gr-qc/0409054 http://arXiv:0603008 [gr-qc] http://arxiv.org/abs/1404.1750

Appendix and etcs.

Graph

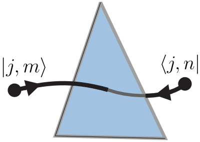
- A quantum tetrahedron and its dual space geometry: the graph.



- A graph γ is : a finite set $\mathcal{N}$ of element n called nodes and a set of $\mathcal{L}$ of oriented couples called links $l = (n, n')$.
- Each node corresponds to one quantum tetrahedron.
- Four links pointing out from the node correspond to each triangle of the tetrahedron.

Variables on single link

- Each link $l = (n, n')$ of the graph is associated with phase space element $(U_{nn'}, J_{nn'}) \in T^*SU(2)$.
- The representation space is the Hilbert space build over $SU(2)$, $\mathcal{H} = L^2[SU(2)]$, one per each link.



variable on 1 link

- The basis is

$$|j,n\rangle\langle j,n'| \in \mathcal{H}_j \otimes \mathcal{H}_j^*,$$

Hilbert space of LQG

- The total Hilbert space of graph γ is : $\mathcal{H}_\gamma = L_2\left[SU(2)^{|\mathcal{L}|}\right]$
- The physical state must satisfies the gauge invariance of $SU(2)$ on each node

$$\psi(U_{nn'}) \to \psi(\lambda_n U_{nn'} \lambda_{n'}^{-1}), \quad \lambda_n \in SU(2)$$

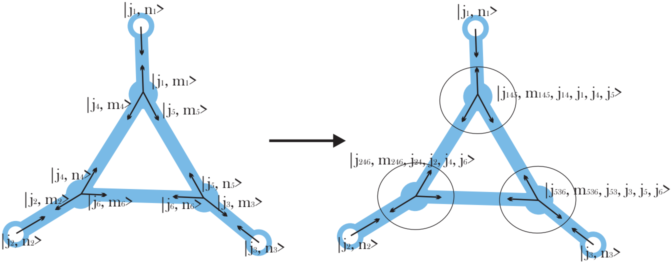
- This can be geometrically interpreted as the closure of the tetrahedron:

$$\sum_{n'} J_{nn'} \left|\psi_{\rm inv}\right\rangle = 0,$$

- This gauge invariance guarantees the quanta to be flat in the interior.

Spin network state in LQG.

- The spin network state of graph γ : product of all state on the links, with gauge invariance condition on each node.



"how to construct spin network basis" : in the end, we take the gauge invariance on each node by setting $j_{145} = j_{246} = j_{536} = 0$.

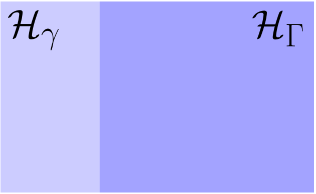
- The physical Hilbert space is $\mathcal{K} = L_2\left[SU(2)^{|\mathcal{L}|}/SU(2)^{|\mathcal{N}|}\right]$, spanned by the spin network state as its basis.

Hilbert space

1. Take the full-non gauge invariant Hilbert space $\mathcal{H}_\gamma$ of the fine graph.

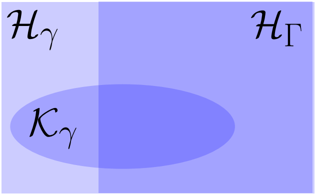
Hilbert space

2. Take the full-non gauge invariant Hilbert space of the subset graph $\mathcal{H}_\Gamma$ by tracing the rest:



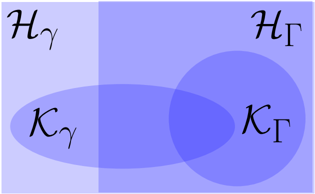
Hilbert space

3. Take the invariant subspace of the fine graph $\mathcal{K}_\gamma$ by tracing projecting with π_n



Hilbert space

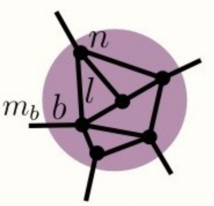
4. Take the invariant subspace of the subset graph $\mathcal{K}_\Gamma$ by tracing projecting with π_N



Then we have the intersection: $\mathcal{K}_\gamma \cap \mathcal{K}_\Gamma$, the density matrix inside the intersection are showed in the result as follow.

Results: Coarse-graining nodes

only showing results:



Coarse graining node.

$$\langle j_b, v|\rho_\Gamma|j_{b'}, v'\rangle = v^{m_b} v'^{m_{b'}} \int dU_b dU_{b'} dU_l D\left(U_b\right)^{j_b}_{m_b n_l} D\left(U_{b'}\right)^{j_{b'}}_{m_{b'} n_l} \overline{\psi\left(U_l, U_b\right)} \psi\left(U_l, U_{b'}\right).$$

Results: Coarse-graining links

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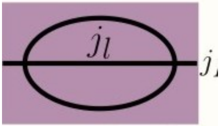

Coarse graining links.

$$\langle J, M, N|\rho_\Gamma|J', M', N'\rangle = \sum_{\alpha,\alpha'} \langle J, M, N, \alpha|\psi\rangle\langle\psi|J', M', N', \alpha'\rangle.$$

Results: Coarse-graining graph

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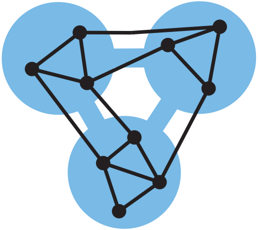

Coarse graining graph.

$$\rho_\Gamma(U_L, U_L') = \sum_{\alpha,\beta} \int dU_l dU_l' \overline{\psi(U_l)} \psi(U_l') D(U_l)^{j_l}_{m_l n_l} D(U_l')^{j'_l}_{m_l' n_l'} i_\alpha^{m_l m_L} i_\beta^{n_l n_L} i_\alpha^{m_l' m_L'} i_\beta^{n_l' n_L'}$$
$$\times D(U_L)^{j_L}_{m_L n_L} D(U_L')^{j'_L}_{m_L' n_L'}$$