First order gravity on the light front

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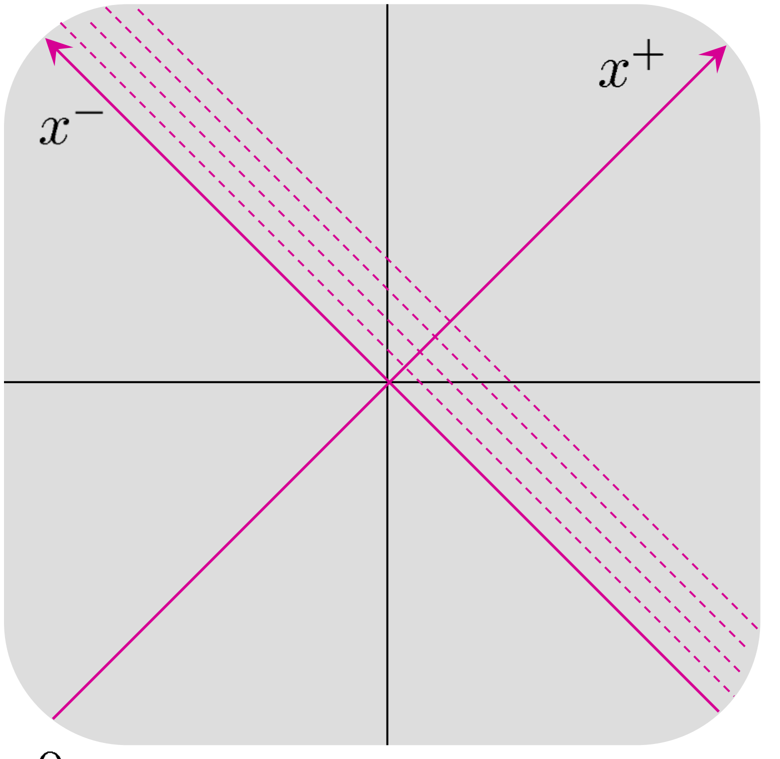
Light Front

Light cone coordinates:

$$x^+ = \frac{1}{\sqrt{2}}(t+x) \qquad x^- = \frac{1}{\sqrt{2}}(t-x)$$



Main features:

- Triviality of the vacuum $(p=0)$

$$p^+ = \frac{m^2 + p_\perp^2}{2p^-} > 0$$

p^- – energy

p^+ – momentum

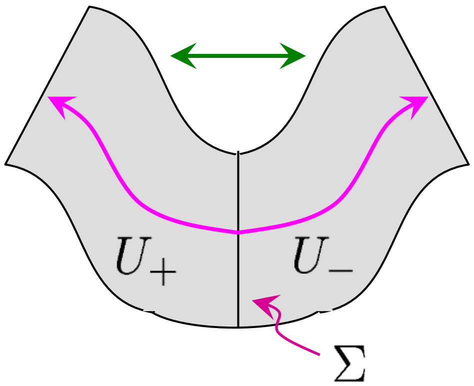
- Non-trivial physics of zero modes $\partial_- \phi_0 = 0$
- Importance of boundary conditions at $x^- \to \pm\infty$
- Presence of second class constraints

$$\int \mathrm{d}^n x \, \partial_+ \phi \partial_- \phi$$

linear in velocities

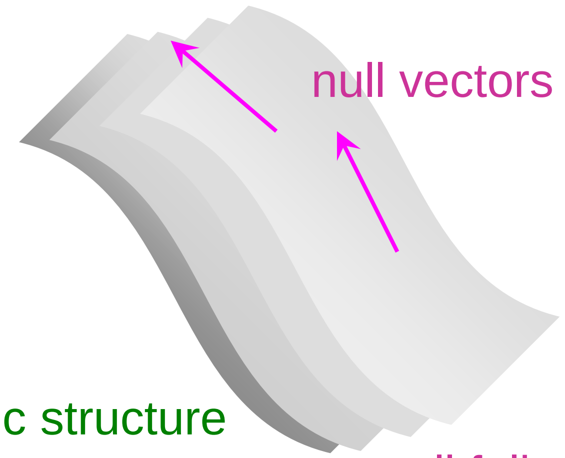
Gravity on the light front

- Sachs(1962) – constraint free formulation



- conformal metrics on $U_\pm$
- intrinsic geometry of Σ
- extrinsic curvature of Σ

- Reisenberger – symplectic structure on the constraint free data



- Torre(1986) – canonical formulation in the metric formalism
- Goldberg,Robinson,Soteriou(1991) _ canonical formulation in
  Inverno,Vickers(1991) the *complex* Ashtekar variables

constraint algebra becomes a *Lie algebra*

We want to analyze the *real* first order formulation on the light front

- constraint free data ➡ exact path integral**?**
- issue of zero modes

- Speziale,Zhang(2013) – null twisted geometries

Technical motivation

In tetrad formalism the null condition can be imposed in the tangent space

3+1 decomposition of the tetrad

$$e^0 = \mathcal{N}\mathrm{d}t + \chi_i E^i_a \mathrm{d}x^a$$
$$e^i = \mathcal{N}^a E^i_a \mathrm{d}t + E^i_a \mathrm{d}x^a$$

$$\mathcal{N} = N + E^i_a \chi_i N^a$$ shift
$$\mathcal{N}^a = N^a + E^a_i \chi^i N$$ lapse

$$g_{\mu\nu} = \begin{pmatrix} -gN^2 + g_{ab}N^aN^b & g_{ab}N^b \\ g_{ab}N^b & g_{ab} \end{pmatrix}$$

$$g_{ab} = \eta_{IJ} e^I_a e^J_b = (\delta_{ij} - \chi_i \chi_j) E^i_a E^j_b$$ spatial metric

Used in various approaches to quantum gravity (covariant LQG, spin foams…)

determines the nature of the foliation

spacelike	lightlike	timelike
$\chi^2 < 1$	$\chi^2 = 1$	$\chi^2 > 1$

light front formulation

Perform canonical analysis for the *real first order* formulation of general relativity on a lightlike foliation

Massless scalar field in 2d

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}x\left((\partial_t\phi)^2 - (\partial_x\phi)^2\right)$$

Solution: $\phi(t,x) = f(x^+) + g(x^-)$

Light front formulation

$$S = \int \mathrm{d}x^+\mathrm{d}x^-\,\partial_+\phi\partial_-\phi$$

$\Rightarrow$ **Primary constraint** $\Phi = \pi - \partial_-\phi \approx 0$

Hamiltonian $H = \int \mathrm{d}x^-\,\lambda\Phi$

$$\left\{\int \mathrm{d}x^-\,\lambda\Phi, \int \mathrm{d}y^-\,\lambda'\Phi\right\} = \int \mathrm{d}x^-\,(\lambda'\partial_-\lambda - \lambda\partial_-\lambda')$$

$\Rightarrow$ Φ is of second class

Stability condition: $\partial_+\Phi = \{\Phi, H\} = -2\partial_-\lambda = 0$

$\Downarrow$

$\lambda = \lambda_0(x^+)$ **zero mode**

$\Phi_0 = \int \Phi\,\mathrm{d}x^-$ first class

Identification:

$$\phi(0,x^-) = g(x^-)$$
$$\lambda_0 = f'(x^+)$$

Conclusions:

- the phase space is one-dimensional
- the lost dimension is encoded in the Lagrange multiplier

Massive theories

$$S = \int \mathrm{d}x^+ \mathrm{d}x^- \left(\partial_+ \phi \partial_- \phi - \frac{m^2}{2} \phi^2 \right)$$

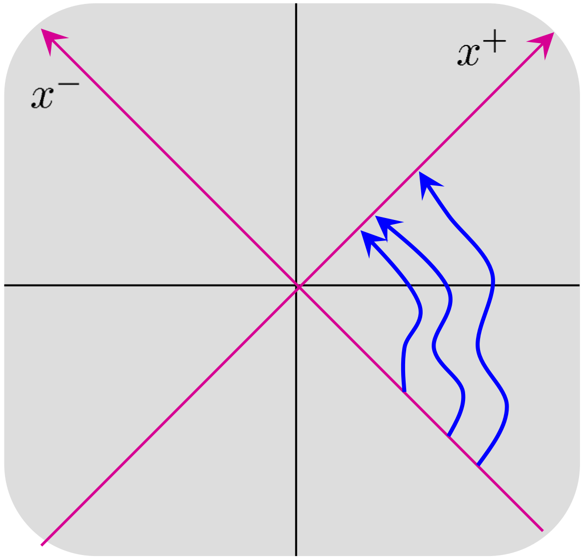
One generates the same constraint but different Hamiltonian

$$\Phi = \pi - \partial_- \phi \approx 0 \qquad H = \int \mathrm{d}x^- \left(\frac{m^2}{2} \phi^2 + \lambda \Phi \right)$$

Stability condition:

$$\partial_+ \Phi = -m^2 \phi - 2\partial_- \lambda = 0$$

inhomogeneous equation



$$\lambda = -\frac{m^2}{2} \int_{-\infty}^{x^-} \phi \mathrm{d}x^- + \boxed{\lambda_0(x^+)}$$

$$\partial_+ \phi = \{\phi, H\} = \lambda$$

The existence of the zero mode contradicts to the natural boundary conditions

In massive theories the light front constraints do not have first class zero modes

In higher dimensions: $m_{\text{eff}}^2 = m^2 + k_\perp^2$ **behave like massive 2d case**

On the light front : **dim. of phase space = num. of deg. of freedom**

First order gravity (spacelike case)

$$S_{\rm HP}[e,\omega]=\frac{1}{4}\int_{\mathcal{M}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\left(F^{KL}(\omega)+\frac{\Lambda}{24}\,e^K\wedge e^L\right)$$

$$e^0=\left(N+E^i_a\chi_i N^a\right)\mathrm{d}t+\chi_i E^i_a\mathrm{d}x^a$$
$$e^i=\left(N^a+E^a_i\chi^i N\right)E^i_a\mathrm{d}t+E^i_a\mathrm{d}x^a$$

Canonical variables:

$$\omega_a^{IJ}\qquad \tilde{P}^a_{IJ}=\frac{1}{4}\,\varepsilon^{abc}\varepsilon_{IJKL}e^K_b e^L_c$$

Fix $x_+=(1,\chi^i)$, $x_+^2<0$ – normal to the foliation

Linear simplicity constraints

$$\Phi^a_I=\varepsilon_{IJ}{}^{KL}x^J_+\tilde{P}^a_{KL}=0$$

Hamiltonian is a linear combination of constraints

$$\Phi^a_I \quad \mathcal{G}_{IJ}=D_a\tilde{P}^a_{IJ}\qquad \mathcal{C}_a=-\tilde{P}^b_{IJ}F^{IJ}_{ab}$$

3

6

$$\mathcal{H}=2\tilde{P}^a_{IK}\tilde{P}^{b,K}_J\left(F^{IJ}_{ab}-\frac{\Lambda}{12}\,\varepsilon_{abc}\varepsilon^{IJKL}\tilde{P}^c_{KL}\right)$$

$$\psi^{ab}=\varepsilon^{IJKL}\tilde{P}^{(a}_{IN}\tilde{P}^{c,N}_J D_c\tilde{P}^{b)}_{KL}$$

secondary constraints

1st class

$$\varepsilon_{IJ}{}^{KL}x^J_+\mathcal{G}_{KL}\quad \mathcal{C}_a\quad \mathcal{H}$$

2d class

$$x^J_+\mathcal{G}_{IJ}\quad \Phi^a_I\quad \psi^{ab}$$

dim. of phase space = 2×18 – 2(3+3+1)-(3+9+6)=4

Hamiltonian analysis on the light front

$$S_{\rm HP}[e,\omega] = \frac{1}{4}\int_{\mathcal{M}} \varepsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}(\omega) + \frac{\Lambda}{24}\, e^K \wedge e^L \right)$$

$$e^0 = \left(N + E^i_a \chi_i N^a\right) \mathrm{d}t + \chi_i E^i_a \mathrm{d}x^a$$

$$e^i = \left(N^a + E^a_i \chi^i N\right) E^i_a \mathrm{d}t + E^i_a \mathrm{d}x^a$$

Canonical variables:

$$\omega^{IJ}_a \qquad \tilde{P}^a_{IJ} = \frac{1}{4}\, \varepsilon^{abc} \varepsilon_{IJKL} e^K_b e^L_c$$

Light front condition $x_+^2 = 0$

Linear simplicity constraints

$$\Phi^a_I = \varepsilon_{IJ}{}^{KL} x^J_+ \tilde{P}^a_{KL} = 0$$

Hamiltonian is a linear combination of constraints

Φ^a_I ⟷ (2) $\mathcal{G}_{IJ} = D_a \tilde{P}^a_{IJ} \qquad \mathcal{C}_a = -\tilde{P}^b_{IJ} F^{IJ}_{ab}$

(7) ↕

$$\mathcal{H} = \frac{1}{2}\, \tilde{p}^a_I \tilde{p}^b_J \left(F^{IJ}_{ab} + \frac{\Lambda}{12}\, \varepsilon_{abc} \varepsilon^{IJKL} \tilde{P}^c_{KL} \right)$$

where $x_- = (-1, \chi^i)$

$$\tilde{p}^a_I = -(x^J_+ - x^J_-) \tilde{P}^a_{IJ} = (0, \tilde{E}^a_i)$$

$$\psi^{ab} = \varepsilon^{IJKL} \tilde{p}^{(a}_I \tilde{p}^c_J D_c \tilde{P}^{b)}_{KL}$$

secondary constraints
+
equation fixing the lapse

The Hamiltonian constraint becomes second class

$$E^a_i \chi^i \left(\partial_a \log N - \omega^{0j}_a \chi_j \right) = 0$$

Tertiary constraints

The crucial observation:

$$\{\psi^{ab},\mathcal{G}_{IJ}\}\approx\{\psi^{ab},\mathcal{C}_a\}\approx 0$$

and

$$\{\psi^{ab},\Phi^c_I\}\sim\mathcal{M}^{ab,cd}=\varepsilon^{(acf}\varepsilon^{b)dg}g_{fg}$$

induced metric on the foliation

has 2 null eigenvectors

Projector on the null eigenvectors

There are two *tertiary constraints*

$$\begin{aligned}\Upsilon_{ab} &= \Pi_{ab,cd}\{\psi^{cd},\mathcal{H}\}\\ &= \frac{1}{2}\Pi_{ab,cd}\varepsilon^{cfg}F^{IJ}_{fg}(\omega)x_{-,I}\tilde{p}^d_J\end{aligned}$$

Stabilization procedure stops due to

$$\det\{\Upsilon,(\Pi\phi)\}\neq 0 \qquad \det\{(\Pi\psi),(\Pi\psi)\}\neq 0$$

Summary

List of constraints:

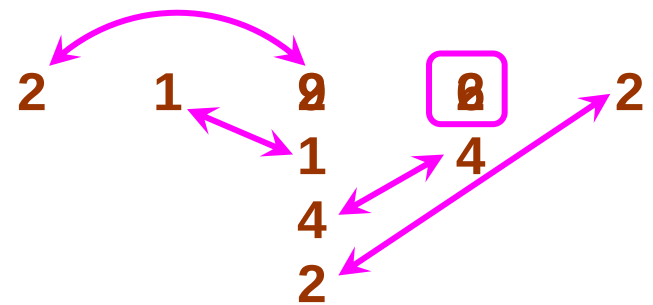
Gauss preserving x_+^I	Gauss rotating x_+^I	spatial diffeos	Hamiltonian	primary simplicity	secondary simplicity	tertiary
$\mathcal{G}^{\parallel}_{IJ}$	$\mathcal{G}^{\perp}_{IJ}$	$\mathcal{C}_a$	$\mathcal{H}$	Φ^a_I	ψ^{ab}	Υ_{ab}

First class

4 3

Lie algebra

Second class



dim. of phase space = 2×18 – 2(4+3)-(2+1+9+6+2)=2

as it should be on the light front

Zero modes

The zero modes of constraints are determined by equations fixing Lagrange multipliers

Potential first class constraints:

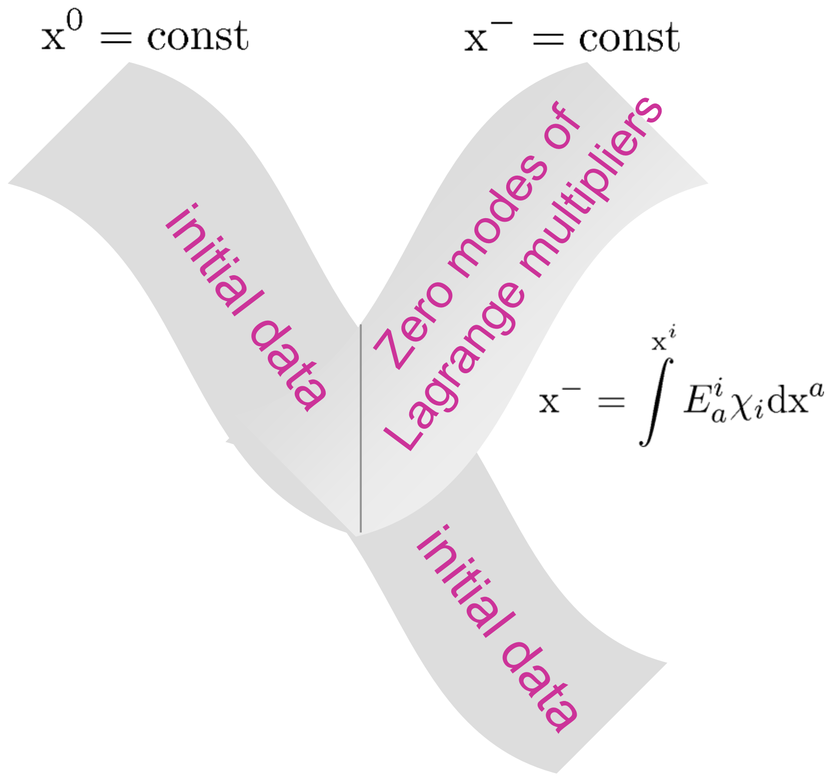
$$\int \mathrm{dx}^- \mathcal{H}$$

$$\int \mathrm{dx}^- \varepsilon_{abc} \tilde{p}^a_I (\tilde{p}^b_J x^J_+) \Phi^c_I$$

$$\int \mathrm{dx}^- (\Pi\phi)_{ab}$$

$$\int \mathrm{dx}^- (\Pi\psi)_{ab}$$

$$\int \mathrm{dx}^- \Upsilon_{ab}$$



$$\tilde{E}^a_i \chi^i \partial_a N = N \tilde{E}^a_i \chi^i \omega^{0j}_a \chi_j$$

$$\partial_a \left(\kappa \tilde{E}^a_i \chi^i \right) = -\kappa \tilde{E}^a_i \chi^i \omega^{0j}_0 \chi_j$$

homogeneous equations

$$\tilde{E}^a_i \chi^i \partial_a \lambda = (\cdots)\lambda + \text{inhomogeneous terms}$$

not expected to generate zero modes

Zero modes

1. $\int \mathrm{dx}^- \mathcal{H}$ time diffemorphisms independent of x^-

Light front condition

$g^{00} = 0$

Gauge transformation

$$\delta g^{00} = \xi^\mu \partial_\mu g^{00} - 2 g^{0\mu} \partial_\mu \xi^0$$
$$= 2N^{-1} E_i^a \chi^i \partial_a \xi^0 = 2N^{-1} \partial_- \xi^0$$

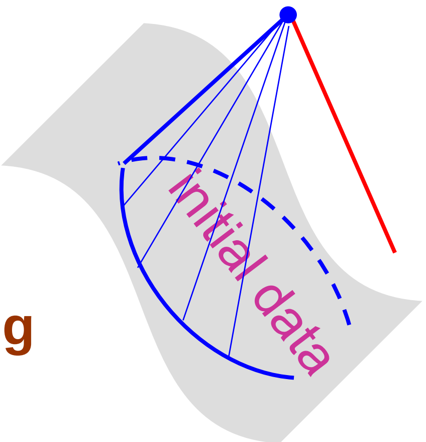
The zero mode corresponds to the residual gauge freedom of the null foliation

2. $\int \mathrm{dx}^- \varepsilon_{abc} \tilde{p}_I^a (\tilde{p}_J^b x_+^J) \Phi_I^c$

would generates shifts of the spin-connection not changing the tetrad

cannot be a gauge generator

There are no (local) zero modes providing additional data needed to fix a solution



Some future directions

- Better understanding of the zero modes?
- What are the appropriate boundary conditions along x^- ?
- Can one solve (at least formally) all constraints?
- What is the right symplectic structure (Dirac bracket)?
- Can this formulation be applied to quantum gravity problems?