Quantum Reduced Loop Gravity

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Look at the inhomogeneous line element in the BKL conjecture Belinski-Khalatnikov-Lifshitz ’70:

\[ ds^2 = N^2(t)dt^2 - e^{2\alpha(t,x)}(e^{2\beta(t,x)})_{ij} \omega^i \otimes \omega^j \]

\( \alpha \) Describes the Volume
\( \beta \) (diagonal matrix, \( \text{Tr} \ \beta = 0 \)) Describes local anisotropies
\( \omega \) one forms corresponding to an homogeneous Bianchi model

**GOAL:**
find a quantum symmetry reduction of LQG compatible with this line element

If we remove the spatial dependence from \( \alpha \) and \( \beta \), we can recover generic Bianchi models
**GOAL:**
Implement on the SU(2) Kinematical Hilbert space of LQG the classical reduction:

\[
A^i_a = c_i(t, x)\omega^i_a \\
E^a_i = p^i(t, x)\omega^a_i
\]

\[
\{p^i(x, t), c_j(y, t)\} = 8\pi G \gamma \delta^i_j \delta^3(x - y)
\]

**First truncation:** we restrict the holonomies to curves along edges \(e_i\) parallel to fiducial \(\omega^a_i\) vectors.

The SU(2) classical holonomies associated to the reduced variables are

\[
R_{h e_i}^j = P(e^{i \int_{e_i} c^i \omega^i_a dx^a(s)\tau_i})
\]

\[
R_{h e_i}^j = \exp (i\alpha^i \tau_i)
\]

Holonomy belong to the U(1) subgroup generated by \(\tau_i\)
Consider fluxes across surfaces $x^a(u,v)$ with normal vectors parallel to the fiducial ones.

The classical reduction implies

$$E_i(S^k) = \int E_i^a \frac{1}{\omega} \omega_a^k dudv = \delta_i^k \int p_i \frac{1}{\omega} dudv$$

For consistency only the diagonal part of the matrix $E_i(S^j)$ is non vanishing.

Second class with the Gauss constraint

$$\chi_i = \sum_{l,k} \epsilon_{il}^k E_k(S^l) = 0$$
How to implement the reduction on the holonomies and consistently impose $\chi_i = 0$?

**Strategy:** Mimic the spinfoam procedure

Impose the **second class constraint weakly** to find a “Physical Hilbert space”

Engle, Pereira, Rovelli, Livine ‘07- ‘08

Imposing a Master constraint strongly on the SU(2) holonomies:

\[ \chi^2 h^j_{e_i} = (8\pi \gamma l_P^2)^2 (\tau^2 - \tau_i \tau_i) h^j_{e_i} = 0 \]

To solve use SU(2) coherent states

\[ |j, \vec{u} > = D^j(\vec{u}) |j, j > = \sum_m |j, m > D^j(\vec{u})_{m j} \]

**Reduced basis Elements**

\[ \langle j, \vec{e}_i | D^j (g) | j, \vec{e}_i \rangle \]
There is a natural way of embedding $U(1)$ cylindrical functions in $SU(2)$ ones:

Projected spinnetworks (Alexandrov, Livine ’02) with the Dupuis-Livine map (Dupuis Livine ’10)

These $SU(2)$ functions have the remarkable property that they are completely determined by their restriction to their $U(1)$ subgroup

$$\tilde{\psi}(g)|_{U(1)} = \psi$$

If we consider projected functions defined over the edge $e_i$ choosing the subgroup $U(1)_i$ as the one generated by $\tau_i$

$$\tilde{\psi}(g)_{e_i} = \sum_{n_i} iD^j_{m=n_i r=n_i} (g) \psi^{n_i}_{e_i}$$

The Master constraint equation selects the degree of the map:

$$|n_i| = j(n)$$

The strong quadratic condition implies the linear one weakly

This is how we find in the $SU(2)$ quantum theory the classical reduction
If we define a Projector $P_{\chi}$ on Physical reduced states:

The projector $P_{\chi}$ acting on $\psi_{\Gamma}$ SU(2) cylindrical functions defined on general Graphs $\Gamma$:

- **Restrict the Graphs** to be part of a cubical lattice

- **Select the states** belonging to the SU(2) subspace where our constraint conditions hold weakly:

$$\tilde{\psi}_i(h) = \sum_{n=-\infty}^{+\infty} \psi^n iD_{n\bar{n}}^{i(n)}(h)$$

What is the fate of the GR constraints?
The reduced states will be of the form:

\[
< h|\Gamma, j_v, x_v >_R = \prod_{v \in \Gamma} \prod_{e \in \Gamma} < j_i, x|j_i, \bar{u}_i > \cdot iD^{j_e} (h_e)_{ji,j_i}
\]

Projection on the intertwiner base of the Livine Speziale Intertwiner: Livine, Speziale ‘07

\[
|j_i, \bar{u}_i > = |j_1, \cdots, j_i, \bar{u}_1, \cdots, \bar{u}_i > = \int dg \prod_i |j_i, \bar{u}_i >
\]

SU(2) intertwiner projected on coherent states: Reduced intertwiner

SU(2) holonomy Projected on coherent states Reduced holonomy
Different Reduced SU(2) intertwiners: inhomogeneities

Different Spin labels: Anisotropies

Homogeneous and anisotropic sector

Homogeneous and Isotropic sector

The Inhomogenous sector
On the reduced space:

Reduced s-knot states

Equivalence class of graphs that preserve the cellular structure:
The regularized Euclidean constraint in the full theory reads:

\[ H^m[N] := \frac{N(n)}{N_m^2} \epsilon^{ijk} \text{Tr} \left[ h^{(m)}_{\alpha ij} h^{(m)}_{sk} \{ h^{(m)}_{sk}^{-1}, V \} \right] \]

We regularize à la Thiemann, but using only elements of the reduced space:

\[ RH^m[N] := \frac{N(n)}{N_m^2} \epsilon^{ijk} \text{Tr} \left[ R h^{(m)}_{\alpha ij} R h^{(m)}_{sk} \{ R h^{(m)}_{sk}^{-1}, V \} \right] \]

Graph-Changing or non-Graph changing version allowed
Action of the operator \((\text{Graph-Changing})\)
on a tri-valent node:

\[
R H^m [N] = \sum_{\text{Permutations}} \left( A_{ij}(j_i, j_j, j_k, m) + A_{ki}(j_i, j_j, j_k, m) \right)
\]

Computed with recoupling theory adapted to the reduced case:
Similar to computation in full theory

Alesci Liegener Thiemann Zipfel
\[ A_{ij}(j_i, j_j, j_k, m) = \]

Remarkably this expression for \( m=1 \) and large values simplify to

\[ \sqrt{j_1 j_2 j_3 + 1} \left[ \{ j_i + m j_j \} \{ j_k + m j_i \} - \{ j_j + m j_i \} \{ j_k + m j_k \} \right] \]

Non trivial solutions to pure gravity
How general is this framework?

Observation: Any 3-metric can be taken to diagonal form by a 3d diffeomorphism, with a residual gauge freedom (reduced diffeomorphisms)

Restriction to cubic lattice can be seen as a gauge fixing at the quantum level of the diffeomorphisms on the 3-metric Alesci, Cianfrani, Rovelli

\[ \eta_{x}^{km} = \delta^{ij} E_{i}(S_{x}^{k}) E_{j}(S_{x}^{m}) = 0, \quad k \neq m, \quad \forall x \in \Sigma \]

\[ \langle \psi|\eta_{x}^{km}|\phi \rangle = 0, \quad k \neq m, \quad \forall x \in \Sigma \]

Weak solution: SU(2) spinnetworks, restricted to reduced graphs with Livine-Speziale intertwiners, synchronized with the frame that diagonalize the metric.

Loop Quantum Gravity in diagonal triad gauge?

Yes but non trivial Hamiltonian (the evolution may not preserve the gauge; in the BKL hypotesis it does) in progress
Semiclassical analysis

\[ \Psi_{\Gamma,H_l}(h_l) = \int \prod_n d g_n \prod_l K_{\alpha_l}(h_l, g_{s(l)} H_l g_{t(l)}^{-1}) \]

\[ H_l = h_l \exp(i \frac{\alpha_l E_l}{8\pi G \hbar c}) \]

Heat Kernel coherent states

SL(2,C) element coding classical data

Hall, Thiemann, Winkler, Sahlmann, Bahr

\[ \Psi_{H_l}(h_l) = \sum_{j_l,i_n} \Psi_{H_l}(j_l, i_n) \Psi_{j_l,i_n}(h_l) \]

intertwiner base
Large distance asymptotic behaviour Bianchi Magliaro Perini

\[
\Psi_{H_l}(h_l) \sim \sum_{j_{l,i_n}} \prod_{l} e^{-\frac{(j_l-j_l^0)^2}{2\sigma_l^2}} e^{-i\xi_l j_l} \prod_{n} \Phi_{i_n} \Psi_{j_l,i_n}(h_l)
\]

- Codes the intrinsic geometry
- Codes the extrinsic curvature
- Livine-Speziale Intertwiners

\[
\dot{j}_0 = \frac{|E|}{8\pi G \hbar \gamma}
\]

\[
\dot{\xi} \sim K = c
\]
Semiclassical states in QRLG

Complexifier:

\[ H_i' = h_i' \exp \left( \frac{\alpha}{8\pi\gamma l_P^2} E_i' \tau_i \right) \]

\[ E_i' \sim p_i \delta_i^2 \]

Area of the smearing surface

Link length

Reduced holonomy:

\[ h_i' = e^{i\theta \tau_i} \in U(1)_i \]

\[ \theta \sim \pm \epsilon_i c_i \]

Sum over magnetic indexes

Heat-Kernel:

\[ K_\alpha(h_l, h_i') = \sum_{m_l = -\infty}^{+\infty} (2j_l + 1)e^{-j_l(j_l+1)\frac{\alpha}{2}} lD_{m_l m_l}(h_l^{-1}h_i') \]

Semiclassical state on the link:

\[ \psi_{H_i'}^\alpha(h_l) = K_\alpha(h_l, H_i') = \sum_{m_l = -\infty}^{\infty} \psi_{H_i'}^\alpha(m_l) lD_{m_l m_l}(h_l^{-1}) \]
\[ \left| \Psi_{H, n^z} \right\rangle_R = R\Psi_{H_1} + R\Psi_{H_2} + R\Psi_{H_3} \approx \text{Gaussians} \]
By a saddle point expansion around the centers of the Gaussians:

\[ \alpha = \frac{1}{(\bar{j})^k} \quad k > 1 \]

\[ \langle \Psi_H \, n^z | R \hat{H}^m_{E_0} | \Psi_H \, n^z \rangle \approx -N(n)C(m)(8\pi \gamma l_P^2)^{3/2} \]

\[ \sum_{\tilde{m}} \sum_{\mu = \pm m} \sum_{\mu_x, \mu_y = \pm m} \sum_{\mu'_x, \mu'_y = \pm m} \sqrt{\tilde{j}_x \tilde{j}_y (\tilde{j}_z + \mu)} \, s(\mu) C_{mm}^{\mu m} \tilde{m} \tilde{m}_0 \]

Classical values

\[ E_i' = 8\pi \gamma l_P^2 \tilde{j}_i \sim \bar{p}^i \delta_i \]
Finally, the expectation value of the scalar constraint is:

\[
\langle \sqrt{R \hat{H}^{1/2}} \rangle_n \approx \frac{1}{\gamma^2} N(n) V(n) \left( \sqrt{\frac{\bar{p}^x \bar{p}^y}{\bar{p}^z}} \sin (\epsilon_{l_x} \bar{c}_x) \sin (\epsilon_{l_y} \bar{c}_y) \epsilon_{l_x} \epsilon_{l_y} + \right.
\]

\[
\left. + \sqrt{\frac{\bar{p}^y \bar{p}^z}{\bar{p}^x}} \sin (\epsilon_{l_y} \bar{c}_y) \sin (\epsilon_{l_z} \bar{c}_z) \epsilon_{l_y} \epsilon_{l_z} + \sqrt{\frac{\bar{p}^z \bar{p}^x}{\bar{p}^y}} \sin (\epsilon_{l_z} \bar{c}_z) \sin (\epsilon_{l_x} \bar{c}_x) \epsilon_{l_z} \epsilon_{l_x} \right)
\]

This expression coincides with the analogous one found in LQC if \( \epsilon_{l_i} = \bar{\mu}_i \).

Sending the regulator to zero

\[
\sqrt{\frac{p^1 p^2}{p^3} c_1 c_2} + \sqrt{\frac{p^2 p^3}{p^1} c_2 c_3} + \sqrt{\frac{p^3 p^1}{p^2} c_3 c_1}
\]

Ashtekar, Wilson-Ewing, Martin-Benito, Mena-Marugan, Pawlowski

Classical Bianchi I Hamiltonian
Add matter: Big Bounce? QFT on quantum spacetime?
Work in progress *Alesci Bilski Cianfrani*

- Study the Physical Hilbert space

- Link to LQC and LQC phenomenology

- Link to Spinfoam Cosmology
  *Bianchi Krajewski Martin-Benito Rennert Rovelli Sloan Vidotto Wilson-Ewing*

- Link to GFT Cosmology
  *Gielen Oriti Sindoni*

- Full theory in a gauge?
- Test the new Hamiltonian: *Alesci Assanioussi Lewandowski*

**Arena for the canonical theory:**
AQG, Master constraint, deparametrized theories.. *Computable!*


THANK YOU