Gravitational Scattering via Twistor Theory

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Work with L. Mason [arXiv:1307.5043, 1308.2820]

Motivation

Why gravity scattering amplitudes?

- Provide important constraints on *any* theory of quantum gravity
- Theoretical 'data'
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c.f., ongoing progress in (planar) gauge theory

Today: tree-level (semi-classical) scattering amplitudes.

Defined by the classical action of a theory:

Definition (Tree-level amplitudes)

Given an action functional $S[\phi]$, non-linear solution to the FEs ('background') ϕ^{cl}, and n solutions $\{\phi_i\}$ to the linearized FEs ('scattering states'), then the *tree-level* scattering amplitude for the $\{\phi_i\}$ on ϕ^{cl} is:

$$\mathcal{M}^0_n(\phi_1, \ldots, \phi_n) = \left. \frac{\partial^n S\left[\phi^{\mathrm{cl}} - \sum_i \epsilon_i \phi_i\right]}{\partial\epsilon_1 \cdots \partial\epsilon_n} \right|_{\epsilon_1 = \cdots \epsilon_n = 0}$$

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- Usually compute by summing Feynman diagrams...*hard!*
- Resulting formulae (drastically) simpler than expected! [DeWitt, Berends-Giele-Kuijf, Mason-Skinner, Nguyen-Spradlin-Volovich-Wen, Hodges, Cachazo-Skinner, Cachazo-He-Yuan, ...]

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Many of these simplifications are related to expressing amplitudes in *twistor theory*.

Questions to Answer:

- Is there some classical action principle giving rise to these simplifications?
- Can we learn anything about the associated twistor geometry?
- Are there *new* expressions for 'amplitudes' in backgrounds that aren't asymptotically flat (e.g., de Sitter space)?

Twistor Toolbox

Basic idea of twistor theory:

$$\text{Physical info on } M \quad \Leftrightarrow \quad \text{Geometric data on } \mathbb{P}\mathscr{T}$$

- $\mathbb{P}\mathscr{T}$ a (deformation of a) 3-dimensional complex projective manifold
- Points $x \in M \leftrightarrow$ holomorphic rational curves $X \subset \mathbb{P}\mathscr{T}$
- $x, y \in M$ null separated iff $X, Y \subset \mathbb{P}\mathscr{T}$ intersect.

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Conformal structure on $M \leftrightarrow \mathbb{C}$-structure on $\mathbb{P}\mathscr{T}$.

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No!

Basic result is [Penrose, Ward] :

Theorem (Non-linear Graviton)

$\exists$ a 1:1 correspondence between:

- *M with self-dual holomorphic conformal structure, and*
- *$\mathbb{P}\mathscr{T}$ with integrable almost complex structure.*

Thing to remember: $\bar{\partial}^2 = 0$ on $\mathbb{P}\mathscr{T} \Leftrightarrow \Psi_{ABCD} = 0$ on M.

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Is there a way to formulate GR as an expansion around the SD sector?

Conformal Gravity

Start with an un-physical theory:

$$
\begin{aligned}
S[g] &= \frac{1}{\varepsilon^2}\int_M \mathrm{d}\mu \; C^{\mu\nu\rho\sigma}\, C_{\mu\nu\rho\sigma} \\
&= \frac{2}{\varepsilon^2}\int_M \mathrm{d}\mu \; \Psi^{ABCD}\, \Psi_{ABCD} \; + \text{top. terms}
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Conformally invariant, with 4^{th}-order equations of motion (non-unitary):

$$(\nabla^A_{A'}\nabla^B_{B'} + \Phi^{AB}_{A'B'})\Psi_{ABCD} = (\nabla^{A'}_A\nabla^{B'}_B + \Phi^{A'B'}_{AB})\widetilde{\Psi}_{A'B'C'D'} = 0$$

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Note: Einstein ($\Phi^{AB}_{A'B'} = 0 = \nabla^{AA'}\Psi_{ABCD}$) and SD/ASD are subsectors of solutions.

Upshot: Perturbative expansion around SD sector. [Berkovits-Witten]

Introduce Lagrange multiplier G_{ABCD}:

$$S[g] \to S[g, G] = \int_M \mathrm{d}\mu \; G^{ABCD}\,\Psi_{ABCD} - \frac{\varepsilon^2}{2}\int_M \mathrm{d}\mu \; G^{ABCD}\,G_{ABCD}$$

Field Equations:

$$\Psi_{ABCD} = \varepsilon^2\, G_{ABCD}, \qquad (\nabla^A_{A'}\nabla^B_{B'} + \Phi^{AB}_{A'B'})G_{ABCD} = 0$$

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ε^2 an expansion parameter around the SD sector.

But we can formulate this in twistor space! [Mason]

Conformal Gravity in Twistor Space

Translation:

$$\Psi_{ABCD} \leftrightarrow N[J] \in \Omega^{0,2}(\mathbb{P}\mathscr{T}, T_{\mathbb{P}\mathscr{T}}), \qquad G_{ABCD} \leftrightarrow b \in \Omega^{1,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4))$$

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Action functional:

$$S[b, J] = \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^3 Z \wedge N \lrcorner b - \frac{\varepsilon^2}{2} \int_{\mathbb{P}\mathscr{T} \times_M \mathbb{P}\mathscr{T}} \mathrm{d}\mu \wedge b_1 \wedge b_2 \, (\sigma_1 \sigma_2)^4$$

Using standard results [Penrose, Atiyah-Hitchin-Singer] :

$$N[J] = \Psi_{ABCD} \sigma^A \, \Sigma^{BC} \frac{\partial}{\partial \sigma_D} , \qquad G_{ABCD} = \int_X \sigma_A \sigma_B \sigma_C \sigma_D \, b|_X$$

Implies FEs on twistor space *equivalent* to those on space-time.

Why do we care?

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Theorem (Anderson, Maldacena)

For M asymptotically de Sitter,

$$S^{\mathrm{CG}}[M] = -\frac{2\,\Lambda^2}{3\varepsilon^2} V_{\mathrm{ren}}(M) = -\frac{\Lambda\,\kappa^2}{3\varepsilon^2} S^{\mathrm{EH}}_{\mathrm{ren}}[M]\,,$$

and asymptotic Einstein states can be singled out in the conformal theory.

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$\Rightarrow$ for tree-level amplitudes,

$$\mathcal{M}^{\mathrm{Ein}} = \frac{1}{\Lambda}\mathcal{M}^{\mathrm{CG}}|_{\mathrm{Ein}}$$

Einstein Gravity in Twistor Space

Einstein degrees of freedom $\Rightarrow$ break conformal invariance

Infinity twistor:

$$I^{\alpha\beta} = \begin{pmatrix} \Lambda\epsilon_{AB} & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}, \qquad I_{\alpha\beta} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \Lambda\epsilon_{A'B'} \end{pmatrix}$$

Obey $I^{\alpha\beta}I_{\beta\gamma} = \Lambda\delta^{\alpha}_{\gamma}$, induce geometric structures:

$$\tau = I_{\alpha\beta}Z^{\alpha}\mathrm{d}Z^{\beta} \in \Omega^{1,0}(\mathbb{P}\mathscr{T}, \mathcal{O}(2)), \qquad \{\cdot,\cdot\} = I^{\alpha\beta}\partial_{\alpha}\,\partial_{\beta}$$

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(Like fixing the conformal factor for metric in Klein representation: $\mathrm{d}s^2 = \frac{\epsilon_{\alpha\beta\gamma\delta}\mathrm{d}X^{\alpha\beta}\,\mathrm{d}X^{\gamma\delta}}{(I_{\alpha\beta}X^{\alpha\beta})^2}$)

Write complex structure on $\mathbb{P}\mathscr{T}$ as finite deformation of 'flat' structure.

Compatability with $I^{\alpha\beta}$, $I_{\alpha\beta}$ $\Rightarrow$

$$\bar{\partial} = \bar{\partial}_0 + I^{\alpha\beta}\partial_\alpha h\,\partial_\beta\,, \qquad h \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$$

$$b \to I_{\alpha\beta}Z^\alpha\,\mathrm{d}Z^\beta\,\tilde{h} = \tau\,\tilde{h} \qquad \tilde{h} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-6))$$

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Action becomes:

$$S[b, J] \to S[\tilde{h}, h] = \Lambda \int_{\mathbb{P}\mathscr{T}} \mathrm{D}^3 Z \wedge \tilde{h} \wedge \left(\bar{\partial}_0 h + \frac{1}{2}\{h, h\}\right) - \frac{\varepsilon^2}{2} \int_{\mathbb{P}\mathscr{T}\times_M \mathbb{P}\mathscr{T}} \mathrm{d}\mu\;\tau_1 \wedge \tau_2 \wedge \tilde{h}_1 \wedge \tilde{h}_2\,(\sigma_1\sigma_2)^4$$

By earlier Theorem, $\Lambda^{-1} S[\tilde{h}, h]$ should compute tree-level Einstein gravity amplitudes. Is this actually true?

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Yes! Lots of technical detail, but upshots are [Adamo-Mason] :

- Second term is generating functional for MHV amplitudes
- Flat space limit = Hodges formulae
- New $\Lambda \neq 0$ formulae
- Apparent MHV formalism induced on twistor space

Further Directions

- Twistor action for Einstein gravity itself?
- More general (i.e., N^kMHV) amplitudes?
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- Momentum space prescription for MHV formalism?

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Thanks!